Unitary groups and uniformisation of curves over a local field (preliminary version)

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Abstract Let K be a complete nonarchimedean local field of characteristic zero having a residue field of odd characteristic. Let $L \supset K$ be the unramified quadratic extension. For a discrete co-compact torsion-free arithmetic subgroup Γ of the group $SU(3, L)$, a one-dimensional rigid analytic space on which Γ acts discretely is constructed. The quotients are projective algebraic curves.

Introduction

In this article we construct a uniformisation of curves by discrete arithmetic subgroups Γ of groups $SU(3, L)$. Here $L \supset K$ is the unramified quadratic extension of a local field K of characteristic zero and residue field k of $char(k) > 2$. The construction consists of two ingredients. First a Γ-invariant subset of the affine building $\mathcal{B}$ of the group $SU(3, L)$ is determined. Then this Γ-invariant subset is used to construct a one-dimensional uniformising space on which Γ acts discretely with proper quotients.

The building $\mathcal{B}$ of the group $SU(3, L)$ is a tree. The Γ-invariant subset is a union of subbuildings (i.e. subtrees) belonging to subgroups $SU(2, L)$ that intersect pairwise in at most a vertex. The vertices in the complement of this subset are such that no two such vertices form an edge.

The uniformising space is constructed by associating to each $SU(2, L)$-building in our Γ-invariant set a rigid analytic variety. This rigid analytic variety is choosen in such a way that $\Gamma \bigcap SU(2, L)$ acts on it discretely with proper quotients. We take a suitable open admissable subspace of each of these rigid varieties and glue them according to the intersections of the

$SU(2,L)$-buildings in our Γ-invariant subset of $\mathcal{B}$. For the glueing to be possible, it is necessary that the group $SU(3,\ell)$ acts on the components of the reduction of the analytical space belonging to a $SU(2,L)$-building that correspond to vertices of type 0 in $\mathcal{B}$. Here ℓ denotes the residue field of L. Finally, a component has to be added for each vertex in $\mathcal{B}$ that is not contained in the Γ-invariant subset. The analytic space we associate to a subbuilding belonging to a group $SU(2,L)$ in our construction is a connected component of Drinfelds étale covering of the p-adic upper halfplane Ω_1. This space has all the properties needed.

Let us compare our construction with the uniformisation of complex curves by arithmetic subgroups of $SL(2,\mathbb{R})$. Then the uniformising space is a hermitian symmetric space. This is the complex unit ball. The quotient of this symmetric space by the arithmetic group is either compact or non-compact, depending on whether the arithmetic group is co-compact or has only finite co-volume. If the quotient is non-compact it can always be compactified.

Over $\mathbb{C}_p$ every algebraic curve has a uniformisation as a quotient of a suitable choosen rigid analytic variety by a discrete subgroup of $SL(2,\mathbb{C}_p)$. The discrete group acts on a tree. Since the building of $SU(3,L)$ is also a tree, it does not seem unreasonable to expect for discrete subgroups of $SU(3,L)$ to act discretely on a rigid analytic variety of dimension one with proper quotient.

If the characteristic of the local field is zero, then all arithmetic subgroups of the linear algebraic group are co-compact (provided they have finite covolume). Therefore the idea of compactifying some symmetric space as in the real case does not make much sense.

We now give a brief outline of the article. The first three sections study the building $\mathcal{B}$ of $SU(3,L)$ and the action of arithmetic groups on $\mathcal{B}$. In §1 we recall some basic definitions and properties of the group $SU(3,L)$ and its building $\mathcal{B}$. In §2 we show that $\mathcal{B}$ can be covered by $SU(2,L)$ buildings in such a way that two such $SU(2,L)$ buildings intersect in at most a vertex. We also define the type of coverings of $\mathcal{B}$ by $SU(2,L)$-buildings we are interested in.

In §3 some examples of arithmetic groups $\Gamma \subset SU(3,L)$, such that a Γ-invariant subset of $\mathcal{B}$ is covered by $SU(2,L)$-buildings having the properties we need, are constructed.

In sections §4 till §7 we study Drinfeld's étale covering of the p-adic upper halfplane Ω_1 in detail. In §4 we construct a connected component Σ of this covering over the field L by glueing affinoids. In §5 we show that an analytical space $\Sigma^{(q-1)}$ that consists of $q-1$ connected components isomorphic to Σ can be defined over the field K. We define an étale map from this space to the p-adic upper halfplane $\Omega_{1,K} := \mathbb{P}^1_K - \mathbb{P}^1(K)$. We also compare the construction with Teitelbaums description of Drinfel'ds étale covering (See [Te]). The étale covering $\Sigma^{(q-1)}$ differs from Drinfel'ds étale covering in the way it is defined over the field K. Drinfel'd works over the maximal unramified extension $K^{nr} \supset K$ and uses a Frobenius map to implicitly define the space over the field K. Therefore our construction uses a different Frobenius map.

In §6 and §7 the space Σ over the field L is constructed by glueing open admissable subsets of Σ together. These admissable subsets are embedded in two projective planes on which the group $SU(2, L)$ acts linearly. The embeddings depend on a discrete torsion-free subgroup $\Gamma \subset SU(2, L)$ and are invariant under the action of this subgroup. Then Σ is obtained by glueing these admissable subsets. As a result one has locally defined coordinates on Σ such that the discrete group $\Gamma \subset SU(2, L)$ acts linearly.

In §8 we recall some properties of the set $Y^s \subset \mathbb{P}^2_L$ consisting of the points in $\mathbb{P}^2_L$ that are stable for all maximal K-split tori in $SU(3, L)$. In particular, we define a $SU(3, L)$-equivariant map from the set Y^s to the building $\mathcal{B}$ of the group $SU(3, L)$. This map is used in §9 to define an open admissable subspace $\Sigma^\circ \subset \Sigma$. The spaces Σ° for suitable subgroups $SU(2, L) \subset SU(3, L)$ can be glued together to form a space on which a co-compact arithmetic group $\Gamma \subset SU(3, L)$ acts discretely. This is done in §10. The quotient by the group Γ is not complete, but it can be compactified. The construction of the uniformising space is analogous to the construction of Σ by glueing together admissable subspaces.

In §11 we speculate on how to generalise the construction to other semisimple algebraic groups. We describe the type of subbuildings $\mathbf{b} \subset \mathcal{B}_G$ that can be used to cover the building $\mathcal{B}_G$ of a semisimple group $G(K)$. Furthermore, a sketch of a construction of an analytic variety on which an arithmetic discrete co-compact subgroup $\Gamma \subset G(K)$ acts with proper and algebraizable quotients is given. I am reasonably convinced that these ideas will work for the K-split symplectic groups and for the unitary groups. We also compare our ideas with compactifications of symmetric spaces over the reals $\mathbb{R}$ and with Shimura varieties. This final section is short on proofs.

1 The group $SU(\mathit{3}, L)$ and its building

1.1. The field. Let $L \supset K$ be as above. We write K° (L°) for the ring of integers of K (L). The residue fields are denoted by k and ℓ. By q we denote the number of elements in the residue field of K. Then q is some power of $p := char(k) > 2$. Let v be the additive valuation on L, normalised such that $v(L^*) = \mathbb{Z}$. The absolute value of x in L is $|x| := q^{-2v(x)}$. We fix an uniformizer π in L°, $v(\pi) = 1$.

1.2. The unitary group. Let $V \cong L^3$ be a vector space equiped with a non-degenerated unitary form h. Then there exists an L-basis $e_1,\ e_0,\ e_2$ of V, such that h has the standard form $h(x, y) = x_1\overline{y}_2 + x_2\overline{y}_1 + x_0\overline{y}_0$. Here $x_1,\ x_0,\ x_2$ are the coordinates of V (or $\mathbb{P}^2_L := \mathbb{P}(V)$) with respect to the basis e_1, e_0, e_2 and $^-$ denotes the action of the nontrivial element of the Galois group $Gal(L/K)$. By $SU(\mathit{3}, L)$ we will mean the group of three by three matrices with coefficients in L that act on V, have determinant one and preserve the form h. Occasionally we will view the group $SU(\mathit{3}, L)$ as the group $G(K)$ of K-rational points of a linear algebraic group G defined over K.

1.3. The building. In the L-module V we introduce the two L^0-submodules $M_0 :=< e_0, e_1, e_2 >$ and $M_1 :=< e_0, \pi e_1, e_2 >$. For a L^0-submodule M of V, we write [M] for the equivalence class $\{\lambda \cdot M| \ \lambda \in L^*\}$.

The building $\mathcal{B}$ of $SU(\mathit{3}, L)$ is the tree, whose vertices are given by the $SU(\mathit{3}, L)$ images of $[M_0]$ and $[M_1]$. The edges (or chambers) are given by the $SU(\mathit{3}, L)$ images of $\{[M_0], [M_1]\}$. Since L/K is unramified, a vertex of type $g([M_0])$ with $g \in SU(\mathit{3}, L)$ is contained in $q^3 + 1$ edges while the other vertices are contained in $q+1$ edges. In this case the tree is not homogeneous.

Let $S \subset G(K)$ be a maximal K-split torus. We may assume that S is the torus that acts diagonally with respect to the basis e_0, e_1, e_2 of V. Let $A \subset \mathcal{B}$ be the apartment associated to the maximal K-split torus $S(K) \cong K^*$. Then S acts on A by translations. The vertices of A are $[M_n], n \in \mathbb{Z}$ where $M_{2n} =< e_0, \pi^n e_1, \pi^{-n} e_2 >$ and $M_{2n+1} =< e_0, \pi^{n+1} e_1, \pi^{-n} e_2 >$. We will often identify the apartment A with the real line $\mathbb{R}$. This will always be done in such a way that the vertices correspond to the set of integers. We will then write n for the vertex $[M_n]$. This identification of the apartment A with the real line $\mathbb{R}$ gives a distance $d_{\mathcal{B}}(a, b) := |a - b|$ on A. This distance can be extended to the entire building $\mathcal{B}$.

1.4. Embedding the building into the building of $SL(3,L)$. The group $SU(3,L)$ is a subgroup $SL(3,L)$ fixed by an involution. Therefore the building $\mathcal{B}$ of the group $SU(3,L)$ can be obtained as the set of points fixed by an involution acting on the building $\mathbb{B}$ of the group $SL(3,L)$.

Let us first recall the description of the building $\mathbb{B}$ in terms of equivalence classes of L°-modules. The vertices of $\mathbb{B}$ are given by the equivalence classes of L°-modules in the L-module $V \cong L^3$. The maximal simplices or chambers in the building are triangles. Three vertices $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{B}$ form a chamber if and only if the corresponding equivalence classes of L°-modules can be represented by modules $M_{\mathbf{v}_i}$, $i = 1,2,3$ that satisfy: $\pi M_{\mathbf{v}_0} \subset M_{\mathbf{v}_1} \subset M_{\mathbf{v}_2} \subset M_{\mathbf{v}_0}$.

Let $M_0 := < e_0, e_1, e_2 >$, $M_1 := < e_0, \pi e_1, e_2 >$ and $M_2 := < \pi e_0, \pi e_1, e_2 >$ be three L°-modules. Then $[M_0], [M_1], [M_2]$ define a triangle in the building $\mathbb{B}$.

Using the unitary form h on $V \cong L^3$ one defines the dual $M^\vee$ of a L°-module M as being the L°-module $M^\vee := \{x \in V \mid \forall (y \in M)\ h(x,y) \in L^\circ\}$. Then the map $M \to M^\vee$ induces an involution τ on the vertices of the building $\mathbb{B}$. The map τ can be extended to the entire building $\mathbb{B}$. Let $\mathbb{B}^\tau \subset \mathbb{B}$ be the set of points that are fixed by the involution τ. Then $\mathbb{B}^\tau$ inherits a simplicial structure from the building $\mathbb{B}$. The simplices of $\mathbb{B}^\tau$ are the intersections of $\mathbb{B}^\tau$ with simplices of the building $\mathbb{B}$ that are non-empty. In fact $\mathbb{B}^\tau$ with this simplicial structure is the building $\mathcal{B}$ of the group $SU(3,L)$.

2 Buildings within buildings

Let $x \in \mathbb{P}^2(L)$ be an L-valued anisotropic point. After rescaling we may assume that either $h(x,x) \in L^\circ - \pi L^\circ$ or that $h(x,x) \in \pi L^\circ - \pi^2 L^\circ$. The stabiliser in $SU(3,L)$ of an L-valued anisotropic point $x \in \mathbb{P}^2_L$ preserves the restriction of the hermitian form h to the line $x^\perp$.

If $h(x,x) \in \pi L^\circ - \pi^2 L^\circ$, then the restriction of h to the line $x^\perp$ is equivalent to the form $y_1\overline{y_1} + \pi y_2\overline{y_2}$. This form does not represent 0 over the field L. In particular, the stabiliser of the point $x \in \mathbb{P}^2(L)$ is a compact subgroup in this case.

If $h(x,x) \in L^\circ - \pi L^\circ$, then the restriction of h to the line $x^\perp$ is equivalent to the form $y_1\overline{y_1} + y_2\overline{y_2}$. This form does represent 0 over L. Hence the stabiliser of x in this case is a subgroup $S(U(1,L) \times U(2,L)) \subset SU(3,L)$. The unitary group $U(2,L)$ acts on the line $x^\perp \subset \mathbb{P}^2_L$ preserving the restriction of the hermitian form h to $x^\perp$ and the group $U(1,L)$ is a finite cyclic group of order $q+1$ acting trivially on $x^\perp$. This group contains a subgroup $SU(2,L)$, which is the connected component of the group (considered as an algebraic group) that contains the identity. Thus we have an embedding of the building $\mathbf{b}$ of the group $SU(2,L)$ stabilising the anisotropic point $x \in \mathbb{P}^2(L)$ into the building $\mathcal{B}$ of $SU(3,L)$.

Every subgroup $SU(2,L) \subset SU(3,L)$ stabilises a unique anisotropic point $x \in \mathbb{P}^2(L)$ such that $h(x,x) \in L^\circ - \pi L^\circ$ after some rescaling. Hence we have a bijection between $SU(2,L)$-buildings $\mathbf{b} \subset \mathcal{B}$ and anisotropic points $x \in \mathbb{P}^2(L)$ such that $h(x,x) \in L^\circ - \pi L^\circ$ after rescaling.

We are interested in coverings $\mathcal{T}$ of $\mathcal{B}$ by buildings of subgroups $SU(2,L)$ such that an edge $\mathbf{e} \in \mathcal{B}$ is contained in at most one building $\mathbf{b} \in \mathcal{T}$. We give definitions of coverings that have this property and cover almost all the building. Furthermore, some examples of (non-arithmetic) discrete co-compact subgroups $\Gamma \subset SU(3,L)$ that preserve such a covering are given.

2.1. Locally transversal systems around a vertex. Let $T_1, T_2 \subset \mathcal{B}$ be trees without extremal vertices. Then T_i is the union of the apartments $A \subset T_i$ for $i = 1,2$. We say that the trees T_1 and T_2 intersect *transversally* if they have no edge in common. The trees T_1 and T_2 are called *transversal* if they intersect transversally.

Let $\mathbf{v} \in \mathcal{B}$ be a vertex of type 0. Let $\mathcal{V}$ be a set of subbuildings $\mathbf{b} \subset \mathcal{B}$ belonging to subgroups $SU(2,L) \subset SU(3,L)$ such that $\mathbf{v} \in \mathbf{b}$ for all $\mathbf{b} \in \mathcal{V}$. We call $\mathcal{V}$ a *locally transversal system around the vertex* $\mathbf{v}$ if and only if for every edge $\mathbf{e} \in \mathcal{B}$ that contains the vertex $\mathbf{v}$ there exists exactly one element $\mathbf{b}_\mathbf{e} \in \mathcal{V}$ such that $\mathbf{e} \in \mathbf{b}_e$.

Let $[M_\mathbf{v}]$ be the equivalence class of L°-modules belonging to the vertex $\mathbf{v}$. Each anisotropic point $\overline{x} \in \mathbb{P}(M_\mathbf{v} \otimes \ell)$ can be lifted to a point $x \in \mathbb{P}^2_L$. The point $x \in \mathbb{P}^2_L$ is stabilized by a unique subgroup $SU(2,L) \subset SU(3,L)$. This subgroup $SU(2,L)$ determines a subbuilding $\mathbf{b} \subset \mathcal{B}$. The edges $\mathbf{e} \in \mathbf{b}$ that contain the vertex $\mathbf{v}$ are uniquely determined by the point $\overline{x} \in \mathbb{P}(M_\mathbf{v} \otimes \ell)$.

We say that a set of anisotropic points $a_1, a_2, \dots, a_s \in \mathbb{P}(M_\mathbf{v} \otimes \ell)$ gives rise to a locally transversal system around the vertex $\mathbf{v}$ if the buildings belonging to a set of lifts of these points to $\mathbb{P}^2_L$ form a locally transversal system around the vertex $\mathbf{v}$.

2.2 Lemma. *Let* $\mathbf{v} \in \mathcal{B}$ *be a vertex of type* 0. *Let* $\overline{h}$ *denote the reduction of the hermitian form* h *on* $M_\mathbf{v} \otimes \ell$. *Let* $x, y \in \mathbb{P}(M_\mathbf{v} \otimes \ell)$ *be two distinct* ℓ*-valued* *anisotropic points. Let $z \in \mathbb{P}(M_{\mathbf{v}} \otimes \ell)$ be the unique point that is orthogonal to both x and y w.r.t. $\overline{h}$. Then the following statements are equivalent:*

i) The restriction of the hermitian form $\overline{h}$ to the line $L_{x,y} :=< x, y >$ is non-degenerated.

ii) The point $z \in \mathbb{P}(M_{\mathbf{v}} \otimes \ell)$ that is orthogonal to both x and y is anisotropic.

iii) $\overline{h}(x,x)\overline{h}(y,y) - \overline{h}(x,y)\overline{h}(y,x) \neq 0$.

iv) The buildings belonging to the groups $SU(2, L)$ that stabilise a lift of x and a lift of y intersect transversally at $\mathbf{v}$.

Proof. The hermitian form $\overline{h}$ is non-degenerated. Hence $\overline{h}(z,z) \neq 0$ if the restriction of $\overline{h}$ to the line $L_{x,y} =< x, y >$ is non-degenerated. If $\overline{h}(z,z) = 0$, then $z \in L_{x,y}$ and the restriction of $\overline{h}$ to the line $L_{x,y} =< x, y >$ is degenerated. Therefore statements (i) and (ii) of the lemma are equivalent.

Let $u \in L_{x,y}$ be the point $u := \overline{h}(y,x)x - \overline{h}(x,x)y$. Then $h(u,x) = \overline{h}(y,x)\overline{h}(x,x) - \overline{h}(x,x)\overline{h}(y,x) = 0$ and $h(u,u) = \overline{h}(x,x) \cdot (\overline{h}(x,x)\overline{h}(y,y) - \overline{h}(y,x)\overline{h}(x,y))$. Therefore the restriction of $\overline{h}$ to the line $L_{x,y} =< x, y >$ is degenerated if and only if $h(u,u) = 0$. Hence statements (i) and (iii) of the lemma are equivalent.

Let G_x and G_y be the stabilisers in the group $SU(3,\ell)$ of the points x and y, respectively. Then the groups G_x and G_y are isomorphic to $S(U(2,\ell) \times U(1,\ell))$. A subgroup $SU(2,\ell)$ of G_x acts on the line $x^{\perp} \subset \mathbb{P}(M_{\mathbf{v}} \otimes \ell)$. Its Borel subgroups are the stabilisers of the ℓ-valued isotropic points in $x^{\perp}$. Hence the subgroups $SU(2,\ell)$ of G_x and G_y have a Borel group in common if and only if the point $z \perp x, y$ is isotropic.

Let $\mathbf{b}_x$ and $\mathbf{b}_y$ be the $SU(2, L)$-buildings belonging to lifts of x and y, respectively. Then the reduction of the stabiliser of the vertex $\mathbf{v}$ in the group $SU(2, L)$ acting on $\mathbf{b}_x$ is the group $SU(2,\ell)$ contained in G_x. Moreover, the reduction of the stabiliser of an edge $\mathbf{e} \ni \mathbf{v}$, $\mathbf{e} \in \mathbf{b}_x$ is a Borel subgroup of the group $SU(2,\ell)$ contained in G_x. The same holds for $\mathbf{b}_y$. Hence the intersection $\mathbf{b}_x \bigcap \mathbf{b}_y$ contains an edge $\mathbf{e} \ni \mathbf{v}$ if and only if the groups $SU(2,\ell)$ in the stabilisers of x and y have a Borel group in common. Hence $\mathbf{b}_x$ and $\mathbf{b}_y$ intersect transversally at $\mathbf{v}$ if and only if the point $z \perp x, y$ is anisotropic. □

2.3 Proposition. *Let $\mathbf{v} \in \mathcal{B}$ be a vertex of type 0. Let $x \in \mathbb{P}(M_{\mathbf{v}} \otimes \ell)$ be an ℓ-valued anisotropic point and let $x^{\perp} \subset \mathbb{P}(M_{\mathbf{v}} \otimes \ell)$ be the line orthogonal* *to the point x. Then the ℓ-valued anisotropic points contained in the line $x^\perp$ together with the point x itself give rise to a locally transversal system around the vertex $\mathbf{v} \in \mathcal{B}$.*

Proof. We first show that the buildings belonging to liftings of distinct ℓ-valued anisotropic points in $x^\perp \cup \{x\}$ are indeed transversal.

Let $e, f \in x^\perp$ be distinct ℓ-valued anisotropic points. Then $< e, f >= x^\perp$ and the unitary form $h \otimes \ell$ is non-degenerated on $x^\perp$. Since the unitary form on $M_{\mathbf{v}} \otimes \ell$ is non-degenerated, the $SU(2, L)$-buildings belonging to liftings of the points e and f are transversal by the lemma above.

Now let us assume that $e \in x^\perp$ and that $f = x$. Then $e \neq f$ and, moreover, e and f are orthogonal for the unitary form $h \otimes \ell$. In particular, the unitary form is non-degenerated on $< e, f >$. Again the $SU(2, L)$-buildings belonging to liftings of the two points e and f are transversal.

The buildings corresponding to liftings of the ℓ-valued anisotropic points in the set $x^\perp \cup \{x\}$ are pairwise transversal. To complete the proof of the proposition it suffices to show that all edges $\mathbf{e} \in \mathcal{B}$ containing the vertex $\mathbf{v}$ are contained in an building that corresponds to a lifting of an anisotropic point contained in $x^\perp \cup \{x\}$. Since the buildings are pairwise transversal, we only have to count the number of edges.

The vertex $\mathbf{v}$ is of type 0 and is contained in $q^3 + 1$ edges of the building $\mathcal{B}$. Each $SU(2, L)$-subbuilding of $\mathcal{B}$ that contains $\mathbf{v}$, contains exactly $q + 1$ edges $\mathbf{e} \ni \mathbf{v}$. Therefore a complete transversal system around the vertex $\mathbf{v}$ consists of $(q^3 + 1)/(q + 1) = q^2 - q + 1$ buildings. The line $x^\perp$ contains $q + 1$ isotropic and $q^2 - q$ anisotropic ℓ-valued points. Therefore the number of anisotropic points contained in the set $x^\perp \cup \{x\}$ equals $q^2 - q + 1$. This concludes the proof of the proposition.

□

2.4. Almost complete transversal systems. A covering $\mathcal{T}$ of $\mathcal{B}$ by trees intersecting transversally is called *complete* if the union of the trees $T \in \mathcal{T}$ equals the entire building $\mathcal{B}$. If $\mathcal{T}$ is complete, then an edge $\mathbf{e} \in \mathcal{B}$ is contained in a unique tree $T \in \mathcal{T}$.

A covering $\mathcal{T}$ of $\mathcal{B}$ by trees intersecting transversally is called *almost complete* if the union of the trees $T \in \mathcal{T}$ contains all vertices $\mathbf{v} \in \mathcal{B}$ of type 0. Let $\mathbf{v}_1 \in \mathcal{B}$ be a vertex of type 1 that is not contained in the union of the trees $T \in \mathcal{T}$. Since all the vertices that form an edge with $\mathbf{v}_1$ are of type 0, they are contained in the union $|\mathcal{T}| := \bigcup\{T \in \mathcal{T}\}$. Therefore only isolated vertices are omitted from the building $\mathcal{B}$.

Let the building $\mathcal{B}$ be embedded into the building $\mathbb{B}$ of the group $SL(3, L)$ as the set of points fixed by an involution. Then we can embed the trees $T \in \mathcal{T}$ in $\mathcal{B} \subset \mathbb{B}$. The convex hull of the trees $T \in \mathcal{T}$ inside the $SL(3, L)$-building equals the $SU(3, L)$-building $\mathcal{B}$. Therefore one can view the building $\mathcal{B}$ as being the compactification inside the $SL(3, L)$-building of the almost complete transversal system $\mathcal{T}$.

We are mainly interested in (almost) complete transversal coverings of $\mathcal{B}$ by $SU(2, L)$-buildings $\mathbf{b} \subset \mathcal{B}$. Below we show their existence and give some examples of non-arithmetic discrete co-compact subgroups $\Gamma \subset SU(3, L)$ admitting a Γ-invariant almost complete transversal system of $SU(2, L)$-buildings.

2.5 Corollary. *Complete transversal coverings of the building $\mathcal{B}$ by subbuildings belonging to subgroups $SU(2, L) \subset SU(3, L)$ exist.*

Proof. We prove this by induction. Start by constructing a complete locally transversal system $\mathcal{V}_{\mathbf{v}}$ of $SU(2, L)$-buildings intersecting in a vertex $\mathbf{v}$. To each building $\mathbf{b} \in \mathcal{V}$ belongs a point $x_{\mathbf{b}} \in \mathbb{P}^2_L$ that is stabilised by the stabiliser in $SU(3, L)$ of $\mathbf{b}$.

In a vertex $\mathbf{v}' \in \mathbf{b}$, $\mathbf{v}' \neq \mathbf{v}$ the $SU(2, L)$-buildings corresponding to lifts of the anisotropic points in the line $x_{\mathbf{b}}^{\perp}$ orthogonal to the reduction of $x_{\mathbf{b}}$ in $\mathbb{P}(M_{\mathbf{v}'} \otimes \ell)$ intersect transversally by the lemma above. Let us call this set $\mathcal{V}_{\mathbf{v}'}$. Then $\mathcal{V}_{\mathbf{v}'} \bigcup \{\mathbf{b}\}$ is a locally complete transversal system around the vertex $\mathbf{v}'$. Furthermore, the union $\mathcal{V}_{\mathbf{v}'} \bigcup \mathcal{V}_{\mathbf{v}}$ is a transversal system.

Repeating this proces one obtains by induction a complete transversal system of $SU(2, L)$-buildings. □

2.6 Proposition. *Let $\Gamma \subset SU(3, L)$ be a discrete torsion-free co-compact subgroup. Then:*

(i) There exists a Γ-invariant complete transversal system of apartments.

(ii) There exists a Γ-invariant complete transversal system of apartments that consists of a single Γ-orbit of apartments.

Proof. Since the characteristic of the residue field is larger than two, each vertex of the building $\mathcal{B}$ is contained in an even number of edges. The same holds for the vertices of the quotient $\mathcal{B}/\Gamma$. One can cover the quotient $\mathcal{B}/\Gamma$ by closed paths such that every edge $\mathbf{e} \in \mathcal{B}/\Gamma$ is traversed only once. The inverse images of these closed paths form a Γ-invariant transversal sytem of apartments covering all of $\mathcal{B}$. This proves statement (i).

To prove statement (ii) one takes a cover of $\mathcal{B}/\Gamma$ by closed paths such that every edge $\mathbf{e} \in \mathcal{B}/\Gamma$ is traversed only once. One can join the paths together such that $\mathcal{B}/\Gamma$ is covered by a single closed path such that every edge $\mathbf{e} \in \mathcal{B}/\Gamma$ is traversed only once. The inverse image of this path is a Γ-invariant complete transversal system that consists of a single Γ-orbit. This proves statement (ii). □

2.7 Example. Let us give an example of a torsion-free discrete co-compact subgroup $\Gamma \subset SU(3, L)$ that leaves invariant a complete transversal covering $\mathcal{T}$ of the building $\mathcal{B}$. Let $\mathbf{v} \in \mathcal{B}$ be a vertex of type 0 and $\mathcal{T}_{\mathbf{v}}$ be a locally transversal system around the vertex $\mathbf{v} \in \mathcal{B}$. For each $SU(2, L)$-building $\mathbf{b} \in \mathcal{T}_{\mathbf{v}}$ we fix a torsion-free discrete co-compact subgroup $\Gamma_{\mathbf{b}} \subset SU(2, L) \cong SL(2, K)$ that acts transitively on the two types of vertices contained in $\mathbf{b}$. Then $\mathbf{b}/\Gamma_{\mathbf{b}}$ consists of two vertices joined by $q + 1$ edges. The group $\Gamma \subset SU(3, L)$ generated by the groups $\Gamma_{\mathbf{b}}$ with $\mathbf{b} \in \mathcal{T}_{\mathbf{v}}$ is a torsion-free discrete cocompact subgroup. The quotient $\mathcal{B}/\Gamma$ consists of $q^2 - q + 1$ vertices of type 1 and a single vertex of type 0. Each vertex of type 1 is joined to the vertex of type 0 by $q + 1$ edges. The subbuildings $\gamma(\mathbf{b})$ with $\gamma \in \Gamma$ and $\mathbf{b} \in \mathcal{T}_{\mathbf{v}}$ form a Γ-invariant complete transversal covering of the building $\mathcal{B}$.

2.8 Example. We construct a discrete co-compact subgroup $\Gamma \subset SU(3, L)$ that preserves an almost complete transversal system of $SU(2, L)$-buildings.

Let $a \in \mathbb{P}^2(L)$ be an isotropic point and let $\mathbf{v}_0 \in \mathcal{B}$ be a vertex of type 0. Let $H_{\mathbf{v}_0,a} \subset \mathcal{B}$ be the half-apartment starting at the vertex $\mathbf{v}_0$ such that the end of $H_{\mathbf{v}_0,a}$ correponds to the point a. Then we take a maximal set Λ of $SU(2, L)$-buildings that contain the half-apartment $H_{\mathbf{v}_0,a}$ and such that for all $\mathbf{b}, \mathbf{b}' \in \Lambda$, $\mathbf{b} \neq \mathbf{b}'$ the intersection $\mathbf{b} \bigcap \mathbf{b}'$ equals $H_{\mathbf{v}_0,a}$. Then the set Λ consists of q^2 distinct $SU(2, L)$-buildings $\mathbf{b}$. Moreover, the union of the $SU(2, L)$-buildings $\mathbf{b} \in \Lambda$ contains all the edges $\mathbf{e} \ni \mathbf{v}_0$.

The set Λ is used to construct a group Γ. Let $H_{\mathbf{b}}$ be the group $SU(2, L) \subset SU(3, L)$ preserving the building $\mathbf{b}$. We fix a building $\mathbf{b}_0 \in \Lambda$. We take a discrete torsion-free subgroup $\Gamma_0 \subset H_{\mathbf{b}_0}$ that acts transitively on both types of vertices of the building $\mathbf{b}_0$. Hence $\mathbf{b}_0/\Gamma_0$ consists of two vertices joined by $q + 1$ edges.

Let $\mathcal{V}$ consist of the vertices $\mathbf{v} \in \mathcal{B}$ that form an edge with the vertex $\mathbf{v}_0$ and that are not contained in the building $\mathbf{b}_0$. The set $\mathcal{V}$ consists of $q^3 - q$ vertices (of type 1). Each vertex $\mathbf{v} \in \mathcal{V}$ is contained in a unique building

$\mathbf{b_v} \in \Lambda$. To a vertex $\mathbf{v} \in \mathcal{V}$ we associate a cyclic subgroup $\Gamma_{\mathbf{v}} \subset H_{\mathbf{b_v}}$ of order $q+1$ that acts transitively on all the edges $e \ni \mathbf{v}$.

Now we define $\Gamma \subset SU(3, L)$ as being the group $\Gamma :=< \Gamma_0, \Gamma_{\mathbf{v}} \mid \mathbf{v} \in \mathcal{V} >$. The group Γ acts transitively on the vertices $\mathbf{v} \in \mathcal{B}$ of type 0 and has q^3-q+1 orbits on the vertices of type 1. An orbit of Γ on vertices of type 1 contains either a unique vertex $\mathbf{v} \in \mathcal{V}$ or it contains all vertices $\mathbf{v} \in \mathbf{b}_0$ of type 1.

For $\mathbf{b} \in \Lambda$, $\mathbf{b} \neq \mathbf{b}_0$ the intersection $\Gamma_{\mathbf{b}} := \Gamma \bigcap H_{\mathbf{b}}$ is generated by the q groups $\Gamma_{\mathbf{v}}$ with $\mathbf{v} \in \mathcal{V} \bigcap \mathbf{b}$. The $\Gamma_{\mathbf{b}}$-images of the edges $\mathbf{e} \ni \mathbf{v}_0$ that contain a vertex $\mathbf{v} \in \mathcal{V} \bigcap \mathbf{b}$ form a regular $(q, q+1)$-tree, which we denote by $\mathcal{T}_{\mathbf{b}}$. The tree $\mathcal{T}_{\mathbf{b}}$ is contained in the building $\mathbf{b}$ and the action of the group $\Gamma_{\mathbf{b}}$ on $\mathcal{T}_{\mathbf{b}}$ is discrete. The quotient $\mathcal{T}_{\mathbf{b}}/\Gamma_{\mathbf{b}}$ consists of a single vertex of type 0 and $q+1$ vertices of type 1.

The set $\mathcal{T} := \{\gamma(\mathbf{b}_0), \gamma(\mathcal{T}_{\mathbf{b}}) \mid \gamma \in \Gamma,\ \mathbf{b} \in \Lambda,\ \mathbf{b} \neq \mathbf{b}_0\}$ is a complete Γ-invariant covering of the building $\mathcal{B}$ by trees that intersect transversally. Let $\mathcal{T}_0 \subset \mathcal{T}$ be the set $\mathcal{T}_0 := \{\gamma(\mathbf{b}_0) \mid \gamma \in \Gamma\}$. Since the group Γ acts transitively on the vertices $\mathbf{v} \in \mathcal{B}$ of type 0, the set $\mathcal{T}_0$ is an almost complete system of $SU(2, L)$-buildings that is Γ-invariant. The set $\mathcal{T}_0$ has the additional property that each vertex $\mathbf{v} \in \mathcal{B}$ of type 0 is contained in a unique building $\mathbf{b} \in \mathcal{T}_0$.

We expect (but cannot prove) that for each group $\Gamma' \subset \Gamma$ of finite index there exists no Γ'-invariant almost complete transversal system of $SU(2, L)$-buildings, except $\mathcal{T}_0$. In particular, we expect that the groups $\Gamma' \subset \Gamma$ of finite index do not admit a Γ'-invariant complete transversal system of $SU(2, L)$-buildings.

3 Arithmetic groups

We consider arithmetic discrete co-compact subgroups of $SU(3, L)$. Over a field of positive characteristic every unitary form of rank > 2 represents 0. Therefore such groups can only exist if $char(K) = 0$.

We give some examples of arithmetic discrete co-compact subgroups $\Gamma \subset SU(3, L)$ that admit an almost complete transversal system of $SU(2, L)$-subbuildings.

3.1. Arithmetic groups. Let $\mathcal{K}$ be a totally real Galois extension of $\mathbb{Q}$ and let $\mathcal{L}$ be a totally imaginary quadratic Galois extension of $\mathcal{K}$. In particular, $\mathcal{L}$ is a CM-field. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{K}$ that is inert in the extension $\mathcal{L} \supset \mathcal{K}$.

Let h_0 be a positive definite hermitian form on $\mathcal{L}^3$. Let $m > 0$ be an integer such that the ideal $\mathfrak{p}^m$ is a principal ideal. Let $s_\mathfrak{p} \in \mathfrak{p}^m$ be a generator of the principal ideal $\mathfrak{p}^m$.

Let $\mathcal{O}_\mathcal{K}$ and $\mathcal{O}_\mathcal{L}$ denote the ring of integers of the field $\mathcal{K}$ and $\mathcal{L}$, respectively. Let $\Lambda \subset \mathcal{L}^3$ be an integer hermitian lattice. Let G_Λ be the algebraic group defined over $\mathcal{O}_\mathcal{K}$ that preserves the lattice Λ and the hermitian form h_0.

Let $\mathcal{K}_\mathfrak{p}$ and $\mathcal{L}_\mathfrak{p}$ denote the completion of $\mathcal{K}$ and $\mathcal{L}$, respectively, w.r.t. the ideal $\mathfrak{p}$

Then $G_\Lambda(\mathcal{O}_\mathcal{L}[1/s_\mathfrak{p}])$ is a discrete co-compact subgroup of the group $SU(3, \mathcal{L}_\mathfrak{p})$. Note that the group does not depend on the choice of the generator $s_\mathfrak{p}$. This is the type of arithmetic groups we consider.

3.2. The special genus. Let $Val(\mathcal{K})$ denote the set of valuations of $\mathcal{K}$. For $\nu \in Val(\mathcal{K})$ we denote by $\mathcal{K}_\nu$ the completion of $\mathcal{K}$ w.r.t. ν. We put $\mathcal{L}_\nu := \mathcal{L} \otimes \mathcal{K}_\nu$.

If the valuation is archimedean, then $\mathcal{K}_\nu \cong \mathbb{R}$ and $\mathcal{L}_\nu \cong \mathbb{C}$, since $\mathcal{L}$ is an imaginary quadratic extension of a totally real field $\mathcal{K}$. If ν corresponds to an inert or ramified prime ideal in the extension $\mathcal{L} \supset \mathcal{K}$, then $\mathcal{L}_\nu$ is equal to to the completion with respect to ν. If the prime ideal corresponding to ν splits, then $\mathcal{L}_\nu$ is equal to the product of the completions of $\mathcal{L}$ w.r.t. the two primes into which the prime ideal corresponding to ν splits.

Let V be the vector space $V := \mathcal{L}^3$ equiped with the positive hermitian form h_0. For the convenience of the reader, we recall the definition of the special genus $gen^\circ(\Lambda)$ of Λ (See [S] definition 1.7):
$gen^\circ(\Lambda) := \{\Lambda' \mid \exists (g \in U(V, h_0))\ \forall(\nu \in Val(\mathcal{K}))\ \exists(h \in SU(V \otimes \mathcal{K}_\nu, h_0))\ \Lambda \otimes \mathcal{O}_{\mathcal{K}_\nu} = g(h(\Lambda' \otimes \mathcal{O}_{\mathcal{K}_\nu}))\}$

3.3 Proposition. *Let $\mathfrak{p} \nmid disc(\Lambda)$. For a vertex $\mathbf{v}$ of type 0 of the building of the group $SU(3, \mathcal{L}_\mathfrak{p})$ we denote by $M_\mathbf{v}$ the $\mathcal{O}_{\mathcal{L}_\mathfrak{p}}$-module corresponding to the vertex.*

i) The lattices $\Lambda_\mathbf{v} := \Lambda[\frac{1}{s_\mathfrak{p}}] \bigcap M_\mathbf{v}$ are contained in $gen^\circ(\Lambda)$ and a representative of every isomorphism class of lattices in $gen^\circ(\Lambda)$ occurs as a lattice $\Lambda_\mathbf{v}$ for some vertex $\mathbf{v}$ of type 0 of the building..

ii) The arithmetic group $G_\Lambda(\mathcal{O}_\mathcal{L}[\frac{1}{s_\mathfrak{p}}])$ has $\sharp gen^\circ(\Lambda)$ orbits on vertices of type 0 in $\mathcal{B}$.

Proof. In [S] the lattices $\Lambda' \in gen^\circ(\Lambda)$ are studied using the neighbourhood method. In particular, the definition of the neighbourhood of a lattice Λ (See [S] definitions 2.1 and 2.3) w.r.t. the prime ideal $\mathfrak{p} \subset \mathcal{L}$ is such that the lattices in the neighbourhood are exactly the integer lattices $\Lambda[\frac{1}{s_\mathfrak{p}}] \bigcap M_\mathbf{v}$ that correspond to the vertices $\mathbf{v} \in \mathcal{B}$ of type $\tau(\mathbf{v}) = 0$.

The isomorphism classes of the lattices in the neighbourhood of the lattice Λ w.r.t. the prime $\mathfrak{p}$ are exactly the isomorphism classes of lattices in $gen^\circ(\Lambda)$ (See [S] theorem 2.10 and the remarks following it).

This shows that statement (i) of the proposition holds. The second statement of the proposition follows directly from statement (i). □

3.4. Minimum norm vectors. Let $N_{\mathcal{K}/\mathbb{Q}} : \mathcal{K} \longrightarrow \mathbb{Q}$ be the norm map of the field $\mathcal{K}$ into the field of rational numbers $\mathbb{Q}$. For a hermitian lattice Λ we denote the minimum norm of a non-zero vector in Λ by $min(\Lambda) := min(\{N_{\mathcal{K}/\mathbb{Q}}(h_0(x,x)) \mid x \in \Lambda, x \neq 0\})$. Let $maxmin(gen^\circ(\Lambda)) := max(\{min(\Lambda') \mid \Lambda' \in gen^\circ(\Lambda)\})$ denote the norm of the longest minimum norm vector of the lattices contained in the special genus of Λ. If $\mathcal{K} = \mathbb{Q}$, then the norm map $N_{\mathcal{K}/\mathbb{Q}}$ is the identity map.

3.5 Proposition. *Let $\Lambda' \in gen^\circ(\Lambda)$ be an integer lattice and let $x, y \in \Lambda'$ be two non-zero vectors such that $y \neq \lambda \cdot x$ for $\lambda \in \mathcal{L}^*$. Let us consider the inert prime ideals $\mathfrak{p}$ such that $h_0(x,x), h_0(y,y) \in \mathcal{O}_\mathcal{K} - \mathfrak{p}\mathcal{O}_\mathcal{K}$. Then the $SU(2, \mathcal{L}_\mathfrak{p})$-buildings $\mathbf{b}_x$ and $\mathbf{b}_y$, belonging to the stabilisers in $SU(3, \mathcal{L}_\mathfrak{p})$ of the points x and y,repectively, intersect transversally for all but finitely many choices of the prime ideal $\mathfrak{p}$.*

Proof. Let us fix vectors $x, y \in \Lambda'$ as in the statement of the proposition. Let $\mathfrak{p} \subset \mathcal{K}$ be a prime ideal that is inert in the extension $\mathcal{L} \supset \mathcal{K}$ and such that $h_0(x,x), h_0(y,y) \in \mathcal{O}_\mathcal{K} - \mathfrak{p}\mathcal{O}_\mathcal{K}$. Then the stabilisers of the vectors $x, y \in \Lambda' \subset V$ in $SU(3, \mathcal{L}_\mathfrak{p})$ contain groups $SU(2, \mathcal{L}_\mathfrak{p})$.

The $\mathcal{O}_\mathcal{L}$-lattice Λ' is the lattice $\Lambda' = \Lambda[\frac{1}{s_\mathfrak{p}}] \bigcap M_\mathbf{v} = \Lambda_\mathbf{v}$ for some vertex $\mathbf{v} \in \mathcal{B}$ of type $\tau(\mathbf{v}) = 0$. Here $M_\mathbf{v}$ is the $\mathcal{L}^\circ_\mathfrak{p}$-module corresponding to the vertex $\mathbf{v}$. Therefore the intersection $\mathbf{b}_x \bigcap \mathbf{b}_y$ of the $SU(2, \mathcal{L}_\mathfrak{p})$-buildings $\mathbf{b}_x$ and $\mathbf{b}_y$ contains the vertex $\mathbf{v} \in \mathcal{B}$ such that $\Lambda_\mathbf{v} = \Lambda'$.

Since the hermitian form h_0 is positive definite on $V = \mathcal{L}^3$, the inequality $h_0(x,x)h_0(y,y) - h_0(x,y)h_0(y,x) \geq 0$ holds. Here equality holds if and only if $x = \lambda \cdot y$ for some $\lambda \in \mathcal{L}^*$. Therefore $h_0(x,x)h_0(y,y) - h_0(x,y)h_0(y,x) > 0$ must hold.

Hence for at most finitely many prime ideals $\mathfrak{p}$ the equality $h_0(x,x)h_0(y,y) - h_0(x,y)h_0(y,x) \equiv 0 \mod \mathfrak{p}$ holds. Using lemma 2.2 (iii), it follows that for all but finitely many choices of the inert prime ideal $\mathfrak{p}$ the buildings $\mathbf{b}_x$ and $\mathbf{b}_y$ intersect transversally. □

3.6 Definition. Let $v_{\mathfrak{p}}$ denote the additive valuation on $\mathcal{K}$ or $\mathcal{L}$ w.r.t. the prime ideal $\mathfrak{p}$, normalised such that $v_{\mathfrak{p}}(\mathcal{K}_{\mathfrak{p}} - \{0\}) = \mathbb{Z}$. For a vector $x \in \Lambda[\frac{1}{s_{\mathfrak{p}}}]$, $x \neq 0$ such that $v_{\mathfrak{p}}(h_0(x,x)) \in 2\mathbb{Z}$, we denote the $SU(2,\mathcal{L}_{\mathfrak{p}})$-building belonging to the stabiliser in $SU(3,\mathcal{L}_{\mathfrak{p}})$ of x by $\mathbf{b}_x$. Let $\mathcal{S} \subset \Lambda[\frac{1}{s_{\mathfrak{p}}}]$ be a set of vectors x such that $v_{\mathfrak{p}}(h_0(x,x)) \in 2\mathbb{Z}$. Then we define $\mathcal{T}_{\mathcal{S}}$ by $\mathcal{T}_{\mathcal{S}} := \{\mathbf{b}_x \mid x \in \mathcal{S}\}$.

3.7 Corollary. *Let $R \geq maxmin(gen^\circ(\Lambda))$ be an integer. For a prime ideal $\mathfrak{p} \subset \mathcal{K}$ that is inert in the extension $\mathcal{L} \supset \mathcal{K}$, we denote by $\mathcal{S}_{\mathfrak{p}} \subset \Lambda[\frac{1}{s_{\mathfrak{p}}}]$ the subset of non-zero vectors such that $h_0(x,x) \in \mathcal{O}_{\mathcal{K}}$ and $N_{\mathcal{K}/\mathbb{Q}}(h_0(x,x)) \leq R$. Let us consider the inert prime ideals $\mathfrak{p}$ such that $h_0(x,x) \notin \mathfrak{p}\mathcal{O}_{\mathcal{K}}$ for all $x \in \mathcal{S}_{\mathfrak{p}}$. Then for all but finitely many choices of the prime ideal $\mathfrak{p}$, the set $\mathcal{T}_{\mathcal{S}_{\mathfrak{p}}}$ is a $G_\Lambda(\mathcal{O}_{\mathcal{L}}[\frac{1}{s_{\mathfrak{p}}}])$-invariant almost complete transversal system of $SU(2,\mathcal{L}_{\mathfrak{p}})$-buildings in the $SU(3,\mathcal{L}_{\mathfrak{p}})$-building.*

Proof. Let us first consider pairs of vectors $x, y \in \mathcal{S}_{\mathfrak{p}}$ that are contained in a fixed $\mathcal{O}_{\mathcal{L}}$-lattice $\Lambda' \in gen^\circ(\Lambda)$. Modulo units of $\mathcal{O}_{\mathcal{L}}$ there are only finitely many such vectors. By the proposition above, for all but finitely many choices of the prime ideal $\mathfrak{p}$ the stabilisers of vectors x and y with $x \neq \lambda \cdot y$, $\lambda \in \mathcal{L}^*$ give rise to $SU(2,\mathcal{L}_{\mathfrak{p}})$-buildings $\mathbf{b}_x$ and $\mathbf{b}_y$ that intersect transversally.

The special genus $gen^\circ(\Lambda)$ contains only finitely many isomorphism classes of $\mathcal{O}_{\mathcal{L}}$-lattices Λ'. For each isomorphism class of lattices, we only have to exclude finitely many prime ideals $\mathfrak{p}$. Hence for all but finitely many inert prime ideals $\mathfrak{p}$ all the $SU(2,\mathcal{L}_{\mathfrak{p}})$-buildings contained in $\mathcal{T}_{\mathcal{S}_{\mathfrak{p}}}$ intersect transversally.

Every vertex $\mathbf{v} \in \mathcal{B}$ of type $\tau(\mathbf{v}) = 0$ is such that the lattice $\Lambda_{\mathbf{v}} := \Lambda[\frac{1}{s_{\mathfrak{p}}}] \bigcap M_{\mathbf{v}}$ is contained in $gen^\circ(\Lambda)$. Since $R \geq maxmin(gen^\circ(\Lambda)$, the lattice $\Lambda_{\mathbf{v}}$ contains at least one non-zero vector $x \in \mathcal{S}_{\mathfrak{p}}$. In particular, the vertex $\mathbf{v}$ is contained in the $SU(2,\mathcal{L}_{\mathfrak{p}})$-building $\mathbf{b}_x \in \mathcal{T}_{\mathcal{S}_{\mathfrak{p}}}$. Therefore $\mathcal{T}_{\mathcal{S}_{\mathfrak{p}}}$ is a $G_\Lambda(\mathcal{O}_{\mathcal{L}}[\frac{1}{s_{\mathfrak{p}}}])$-invariant almost complete transversal system of $SU(2,\mathcal{L}_{\mathfrak{p}})$-buildings. □

3.8 Lemma. *There exists a primitive vector $x_0 \in \Lambda$ such that the subset $\{\Lambda' \in gen^\circ(\Lambda) \mid x_0 \in \Lambda'\} \subset gen^\circ(\Lambda)$ contains a representative of each isomorphism class of lattices that is contained in $gen^\circ(\Lambda)$.*

Proof. Let us fix an inert prime $\mathfrak{p}$. There exists an apartment $A \ni \mathbf{v}_0$ in the building $\mathcal{B}$ of $SU(3, \mathcal{L}_\mathfrak{p})$ such that the set of lattices $\{\Lambda_\mathbf{v} = M_\mathbf{v} \bigcap \Lambda[\frac{1}{s_\mathfrak{p}}] \mid \mathbf{v} \in A,\ \tau(\mathbf{v}) = 0\}$ contains a representative of each isomorphism class contained $gen^\circ(\Lambda)$. The apartment A can be choosen in such a way that the torus T that corresponds to apartment A is defined over the number field $\mathcal{K}$. The torus T that acts on A then fixes a vector $x \in \Lambda$. We may choose the vector x to be primitive. Hence $x_0 = x$ satisfies the lemma. □

3.9 Proposition. *Let $x_0 \in \Lambda$ be as in the lemma above. Let us denote by $\mathcal{S}_{x_0,\mathfrak{p}} \subset \Lambda[\frac{1}{s_\mathfrak{p}}]$ the subset $\mathcal{S}_{x_0,\mathfrak{p}} := G_\Lambda(\mathcal{O}_\mathcal{L}[\frac{1}{s_\mathfrak{p}}]) \cdot x_0 \subset \Lambda[\frac{1}{s_\mathfrak{p}}]$. Then the following two statements hold:*

i) For almost all inert primes $\mathfrak{p}$ the set $\mathcal{T}_{\mathcal{S}_{x_0,\mathfrak{p}}}$ is well-defined.

ii) For almost all inert primes $\mathfrak{p}$ the set $\mathcal{T}_{\mathcal{S}_{x_0,\mathfrak{p}}}$ is a $G_\Lambda(\mathcal{O}_\mathcal{L}[\frac{1}{s_\mathfrak{p}}])$-invariant almost complete transversal system of $SU(2, \mathcal{L}_\mathfrak{p})$-buildings.

Proof. The set $\mathcal{T}_{\mathcal{S}_{x_0,\mathfrak{p}}}$ is well-defined if and only if the stabiliser in $SU(3, \mathcal{L}_\mathfrak{p})$ of each vector that is contained in the set $\mathcal{S}_{\mathcal{S}_{x_0,\mathfrak{p}}} \subset \Lambda[\frac{1}{s_\mathfrak{p}}]$ contains a group $SU(2, \mathcal{L}_\mathfrak{p})$. Hence the set $\mathcal{T}_{\mathcal{S}_{x_0,\mathfrak{p}}}$ is well-defined for any inert prime ideal $\mathfrak{p}$ such that $h_0(x_0, x_0) \notin \mathfrak{p}$. Therefore statement (i) of the proposition holds.

Each lattice $\Lambda_\mathbf{v} := M_\mathbf{v} \bigcap \Lambda[\frac{1}{s_\mathfrak{p}}]$ with $\mathbf{v} \in \mathcal{B}$ of type $\tau(\mathbf{v}) = 0$ contains only finitely many elements of $\mathcal{S}_{x_0,\mathfrak{p}}$. The $SU(2, \mathcal{L}_\mathfrak{p})$-buildings $\mathbf{b} \in \mathcal{T}_{\mathcal{S}_{x_0,\mathfrak{p}}}$ with $\mathbf{b} \ni \mathbf{v}$ intersect transversally if and only if for all vectors $x, y \in \mathcal{S}_{x_0,\mathfrak{p}} \bigcap \Lambda_\mathbf{v}$, $x \neq \lambda \cdot y,\ \lambda \in \mathcal{L}^*$ the inequality $h_0(x,x) \cdot h_0(y,y) - h_0(x,y) \cdot h_0(y,x) \not\equiv 0 \mod \mathfrak{p}$ holds. Therefore the second statement of the proposition follows from the finiteness of the intersections $\mathcal{S}_{x_0,\mathfrak{p}} \bigcap \Lambda_\mathbf{v}$ and the finiteness of the number of isomorphism classes in $gen^\circ(\Lambda)$. □

3.10 Example. Let $\mathcal{K} := \mathbb{Q}$ and let $\mathcal{L} := \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic extension of $\mathcal{K}$ with $d \in \mathbb{Z}_{<0}$ square-free. We take $\Lambda \cong \mathcal{O}_\mathcal{L}^3$ in $\mathcal{L}^3$ with the standard hermitian form $h_0(x,y) = x_0\overline{y_0} + x_1\overline{y_1} + x_2\overline{y_2}$. Let p be a prime that is inert in the extension $\mathcal{L} \supset \mathcal{K}$.

The unimodular hermitian lattices Λ are well-studied for small values of d. For $d = -1, -2, -3, -11$ each lattice in the special genus $gen^\circ(\Lambda)$ contains minimum norm vectors of length 1. The number of non-isomorphic lattices in $gen^\circ(\Lambda)$ equals 1 if $d = -1, -3$ and $\sharp gen^\circ(\Lambda) = 2$ if $d = -2, -11$ (See [S] table 1 and [H] table 1).

Let $\mathcal{S}_p := \{x \in \Lambda \mid h_0(x,x) = p^{2n}, n \in \mathbb{Z}_{\geq 0}\}$. Then $\mathcal{T}_{\mathcal{S}_p}$ is an $G_\Lambda(\mathcal{O}_\mathcal{L}[\frac{1}{p}])$-invariant almost complete transversal system of $SU(2, \mathcal{L}_\mathfrak{p})$-buildings.

In general, the transversal system is not complete. The only exception occurs, when $p = 2$ and $d = -3$. Then the number of vertices $\mathbf{v}$ of type $\tau(\mathbf{v}) = 1$ that are neighbours of a fixed vertex $\mathbf{v}_0$ of type $\tau(\mathbf{v}_0) = 0$ equals $2^3 + 1 = 9$. Furthermore, the vertex $\mathbf{v}_0$ is contained in three $SU(2, \mathcal{L}_\mathfrak{p})$-buildings $\mathbf{b} \in \mathcal{S}_2$ and hence all the 9 vertices that are neighbours of $\mathbf{v}_0$ are contained in a building $\mathbf{b} \in \mathcal{T}_\mathcal{S}$. Since the group $G_\Lambda(\mathcal{O}_\mathcal{L}[\frac{1}{2}])$ acts transitively on vertices of type 0, it follows that $\mathcal{T}_{\mathcal{S}_2}$ is complete.

To conclude this section we describe in the proposition below the relation between the lattices for the group $\Gamma \bigcap S(U(1,L) \times U(2,L))$ and the lattices for $\Gamma \subset SU(3,L)$. Here $\Lambda^\vee$ denotes the dual lattice $\Lambda^\vee := \{x \in \Lambda \otimes \mathcal{L} \mid (\forall y \in \Lambda) h_0(x,y) \in \mathcal{O}_\mathcal{K}\}$.

3.11 Proposition. *Let* $\mathbf{v} \in \mathcal{B}$ *be a vertex of type* $\tau(\mathbf{v}) = 0$ *and let* $x \in \Lambda_\mathbf{v}$ *be a primitive vector such that* $h_0(x,x) \notin \mathfrak{p}$. *Let* $H \cong S(U(1,L) \times U(2,L)) \subset SU(3,L)$ *be the stabiliser of the vector* x. *Let* $\mathbf{b}_x \subset \mathcal{B}$ *be the building of* H *and let* $\Lambda_{\mathbf{v},H} \subset \Lambda_\mathbf{v}$ *be the lattice* $\Lambda_{\mathbf{v},H} := (< x > \bigcap \Lambda_\mathbf{v}) \oplus (x^\perp \bigcap \Lambda_\mathbf{v})$. *Then the following statements hold:*

i) $(< x > \bigcap \Lambda_\mathbf{v})^\vee / (< x > \bigcap \Lambda_\mathbf{v}) \cong (x^\perp \bigcap \Lambda_\mathbf{v})^\vee / (x^\perp \bigcap \Lambda_\mathbf{v}) \cong \Lambda_\mathbf{v} / \Lambda_{\mathbf{v},H}$.

ii) $\mathbf{v} \in \mathbf{b}_x \Leftrightarrow h_0(x,x) \notin \mathfrak{p}$.

Proof. Since $x \in \Lambda_\mathbf{v}$, the group $\Gamma \bigcap H \subset H$ is co-compact. Therefore, the lattice $\Lambda_{\mathbf{v},H} \subset \Lambda_\mathbf{v}$ has finite index. Let $y_i \in \Lambda_\mathbf{v}$, $i = 1, \ldots, n$ be representatives of the classes of $\Lambda_\mathbf{v}/\Lambda_{\mathbf{v},H}$. By construction the dual of $\Lambda_{\mathbf{v},H}$ is $\Lambda^\vee_{\mathbf{v},H} = (< x > \bigcap \Lambda_\mathbf{v})^\vee \oplus (x^\perp \bigcap \Lambda_\mathbf{v})^\vee$. Therefore there exist elements $a_i \in (< x > \bigcap \Lambda_\mathbf{v})^\vee$ and elements $b_i \in (x^\perp \bigcap \Lambda_\mathbf{v})^\vee$, such that $y_i = a_i + b_i$ for $i = 1, \ldots, n$.

If $a_i - a_j \in (< x > \bigcap \Lambda_\mathbf{v})$ for $i \neq j$, then $b_i - b_j \in \Lambda_\mathbf{v}$. Therefore $b_i - b_j \in (x^\perp \bigcap \Lambda_\mathbf{v})$ and $y_i - y_j \in \Lambda_{\mathbf{v},H}$. Similarly, if $b_i - b_j \in (x^\perp \bigcap \Lambda_\mathbf{v})$ for $i \neq j$, then $a_i - a_j \in (< x > \bigcap \Lambda_\mathbf{v})$ and $y_i - y_j \in \Lambda_{\mathbf{v},H}$.

Therefore $a_i \not\equiv a_j \mod (< x > \bigcap \Lambda_\mathbf{v})$ and $b_i \not\equiv b_j \mod (x^\perp \bigcap \Lambda_\mathbf{v})$ for all $1 \leq i,j \leq n$, $i \neq j$. Let $m_x := \sharp(< x > \bigcap \Lambda_\mathbf{v})^\vee / (< x > \bigcap \Lambda_\mathbf{v})$ and let

$m_{x^\perp} := (x^\perp \bigcap \Lambda_{\mathbf{v}})^\vee/(x^\perp \bigcap \Lambda_{\mathbf{v}})$. Then $\sharp\Lambda^\vee_{\mathbf{v},H}/\Lambda_{\mathbf{v},H} = m_x \cdot m_{x^\perp}$. Since the representatives $y_i = a_i + b_i$ of $\Lambda_{\mathbf{v}}/\Lambda_{\mathbf{v},H}$ are such that both $a_i \mod (x^\perp \bigcap \Lambda_{\mathbf{v}})$ and $b_i \mod (x^\perp \bigcap \Lambda_{\mathbf{v}})$ are distinct for all $i = 1, \ldots, n$, $\sharp\Lambda_{\mathbf{v}}/\Lambda_{\mathbf{v},H} \leq min(m_x, m_{x^\perp})$ holds. Since the lattice $\Lambda_{\mathbf{v}}$ is unimodular, the equality $min(m_x, m_{x^\perp})^2 = m_x \cdot m_{x^\perp}$ must hold. This is only possible if $m_x = m_{x^\perp}$. Therefore the vectors a_i, $i = 1, \ldots, n$ represent all the classes of $(x^\perp \bigcap \Lambda_{\mathbf{v}})^\vee/(x^\perp \bigcap \Lambda_{\mathbf{v}})$ and the vectors b_i, $i = 1, \ldots, n$ represent all the classes of $(x^\perp \bigcap \Lambda_{\mathbf{v}})^\vee/(x^\perp \bigcap \Lambda_{\mathbf{v}})$. From this statement (i) of the proposition follows.

The second statement of the proposition follows from the fact that the group H is the stabiliser of a vector in $f \in M_{\mathbf{v}}$ such that $h(f, f) = 1$. □

4 An equivariant étale covering of the p-adic upper half plane

A detailed description of a certain étale covering of the p-adic upper half plane Ω_1 is given. Instead of considering the group $SL(2, K)$ acting on $\mathbb{P}^1_K$, we consider the isomorphic group $SU(2, L)$ acting on $\mathbb{P}^1_L$ preserving a unitary form h_2. Then $\Omega_1 \subset \mathbb{P}^1_L$ is obtained by omitting the L-valued isotropic points from the projective line.

A finite covering Σ of the p-adic upper half plane $\Omega_1 := \mathbb{P}^1_L - \{x \in \mathbb{P}^1(L) \mid h_2(x, x) = 0\}$ is constructed by glueing affinoids. We give a pure affinoid covering of the rigid analytic variety Σ and define an action of the group $SU(2, L)$ on it. The covering Σ is $SU(2, L)$-equivariant finite étale of degree $q + 1$.

4.1. The building. Let us define the building $\mathbf{b}$ of $SU(2, L)$ using equivalence classes of L°-modules in L^2 equiped with the unitary form h_2. All equivalence classes of L°-modules in L^2 give the building of $SL(2, L)$. The building of the group $SU(2, L)$ is now given by the equivalence classes of L°-modules that have a basis consisting of isotropic vectors. Let $e_1, e_2 \in L^2$ be two isotropic vectors, such that the unitary form is given by $h_2(x, y) = x_1\overline{y}_2 + x_2\overline{y}_1$. Let $M_0 =< e_1, e_2 >$ and let $M_1 =< e_1, \pi^{-1}e_2 >$. Then the vertices of the building of $SU(2, L)$ are the $SU(2, L)$-images of the equivalence classes $[M_0]$ and $[M_1]$. The edges of the building are give by the images of $\{[M_0], [M_1]\}$.

Each L°-module is equiped with a unitary form coming from the form h_2. On M_0 the unitary form is non-degenerated, whereas on M_1 it is degenerated.

Let $M^\vee$ be the dual module $M^\vee = \{x \in L^2 \mid \forall (y \in M)\ h_2(x,y) \in L^\circ\}$. Then $M_0^\vee = M_0$ and $M_1^\vee = \pi M_1$. The vertices of the building of $SU(2,L)$ are precisely the equivalence classes of L°-modules $[M]$ such that $[M^\vee] = [M]$ holds. Hence the $SU(2,L)$-building can be seen as the set of points fixed by an involution acting on the building of the group $SL(2,L)$.

On the building $\mathbf{b}$ an $SU(2,L)$-equivariant distance function $d_{\mathbf{b}}(-,-)$ exists. We normalise it such that $d_{\mathbf{b}}(\mathbf{v},\mathbf{v}') = 1$ for vertices $\mathbf{v},\mathbf{v}'$ that form an edge in the building.

4.2. The p-adic upper half plane. Let $\mathbf{b}$ denote the building of $SU(2,L)$. We briefly recall the standard pure affinoid covering of Ω_1. To the standard edge $\mathbf{e}_0 \in \mathbf{b}$ we associate the affinoid space $X_{\mathbf{e}_0}^{\Omega_1} \subset \Omega_1$ given by:
$1 \geq |\frac{x_1}{x_2}| \geq |\pi|$, $|\frac{x_1}{x_2} - c| = 1$, $|\frac{\pi x_2}{x_1} - c| = 1$, $\forall c \in (L^\circ)^*$ such that $c + \overline{c} = 0$.
To the vertices $\mathbf{v}_0$ and $\mathbf{v}_1$ of the edge $\mathbf{e}_0 \in \mathbf{b}$ we associate the open affinoid subspaces $X_{\mathbf{v}_0}^{\Omega_1}$ and $X_{\mathbf{v}_1}^{\Omega_1}$ of $X_{\mathbf{e}_0}^{\Omega_1}$ given by: $|\frac{x_1}{x_2}| = 1$ and $|\frac{\pi x_2}{x_1}| = 1$, respectively. Let $\mathcal{A}(X_{\mathbf{e}}^{\Omega_1})$ and $\mathcal{A}(X_{\mathbf{v}}^{\Omega_1})$ denote the affinoid algebras corresponding to the affinoid spaces $X_{\mathbf{e}}^{\Omega_1}$ and $X_{\mathbf{v}}^{\Omega_1}$, respectively.

One has a $SU(2,L)$-equivariant map $\psi : \Omega_1 \longrightarrow \mathbf{b}$. For $x \in X_{\mathbf{e}_0}^{\Omega_1}$ one puts $\psi(x) := (1 - v(\frac{x_1}{x_2})) \cdot \mathbf{v}_0 + v(\frac{x_1}{x_2}) \cdot \mathbf{v}_1$. Since $|\frac{g^* x_i}{x_i}| = 1$, $i = 1,2$, for all $x \in X_{\mathbf{e}_0}^{\Omega_1}$ and all $g \in P_{\mathbf{e}_0}$, this map does not depend on the choice of the coordinates x_i, $i = 1,2$. Let $x \in \Omega_1$ be a point. There exists an element $g \in SU(2,L)$ such that $g(x) \in X_{\mathbf{e}_0}^{\Omega_1}$. In this situation we put $\psi(x) = g^{-1}(\psi(g(x)))$. Then the function ψ is well-defined and $SU(2,L)$-equivariant.

4.3. Galois action. The analytical variety Ω_1 is defined over the field L. Therefore the Galois group $Gal(L/K)$ acts on Ω_1. To make the Galois action explicit, we embed Ω_1 into a product of two projective lines. Let $\mathbb{P}_L^1 \times \mathbb{P}_L^1$ be the product of two projective lines with coordinates $(x_1,\ x_2)$ and $(z_1,\ z_2)$, respectively. On it we take a quadratic form $x_1 z_2 + x_2 z_1$. The action of the group $SU(2,L)$ is such that it preserves the quadratic form. An element $g \in SU(2,L)$ that acts on the coordinates x_i through a two by two matrix $M(g)$, acts on the coordinates z_i through the matrix $\overline{M(g)}$, obtained by replacing each coefficient $m_{i,j}$ of the matrix $M(g)$ by the Galois conjugate $\overline{m_{i,j}}$.

Let Ω_1 be contained in the projective line $\mathbb{P}_L^1$ defined by $x_1 z_2 + x_2 z_1 = 0$. The interchange of the coordinates x_i and z_i, $i = 1,2$ is a combination of the action of the non-trivial element of the Galois group $Gal(L/K)$ and the diagonal element $diag(1,-1)$. Therefore (x_1, x_2) and $(\overline{z_1}, -\overline{z_2})$ denote the same point of Ω_1. One can also use the coordinates z_i, $i = 1, 2$ on Ω_1, to define a $SU(2, L)$-equivariant map $\psi^\vee : \Omega_1 \to \mathbf{b}$. Of course, $\psi^\vee((z_1, z_2)) = \psi((x_1, x_2))$, if (z_1, z_2) and (x_1, x_2) denote the same point in Ω_1.

Let $g_{\mathbf{e}_0} \in GU(2, L)$ be an element that permutes the two vertices of the building that are contained in the edge $\mathbf{e}_0$. We can choose the element $g_{\mathbf{e}_0}$ in such a way that $g^*_{\mathbf{e}_0} \frac{x_1}{x_2} = -\frac{\pi x_2}{x_1}$ holds. Without changing the action of g_{e_0} on the points in Ω_1, we can redefine the action on coordinates as being given by $(x_1, x_2) \to (\pi z_2, z_1)$ and $(z_1, z_2) \to (\pi x_2, x_1)$. The action of the element $g_{\mathbf{e}_0}$ now incorporates the action on Ω_1 of the non-trivial element of the Galois group $Gal(L/K)$.

An element $g \in GU(2, L)$ such that $v(\det(g)) \equiv 1 \mod 2\mathbb{Z}$ acts as a Galois element through permutation of the x_i and z_i coordinates on Ω_1. Indeed, one uses the fact that $g = h \cdot g_{\mathbf{e}_0}$ with $v(\det(h)) \equiv 0 \mod 2\mathbb{Z}$.

4.4. The étale covering. We define a covering of degree $q+1$ of the affinoids $X^{\Omega_1}_{\mathbf{e}}, X^{\Omega_1}_{\mathbf{v}} \subset \Omega_1$. Let $f_{\mathbf{e}_0} \in \mathcal{A}(X^{\Omega_1}_{\mathbf{e}_0})$ be the following function: $f_{\mathbf{e}_0}(x) := \frac{x_1}{x_2} \cdot \frac{1+(\frac{x_1}{x_2})^{(q-1)}}{1+(-\frac{\pi \cdot x_2}{x_1})^{(q-1)}}$. The element $g_{\mathbf{e}_0} \in GU(2, L)$ that permutes the two vertices contained in the edge $\mathbf{e}_0$ acts on $f_{\mathbf{e}_0}$ as $g^*_{\mathbf{e}_0} f_{\mathbf{e}_0} = -\pi / f_{\mathbf{e}_0}$. Let $h_{\mathbf{e}_0}$ be a $q+1$-th root of the function $-f_{\mathbf{e}_0}$ and let $h^\vee_{\mathbf{e}_0}$ be a $q+1$-th root of the function $\pi / f_{\mathbf{e}_0}$. By $\mathcal{A}_\Sigma$ we denote the affinoid algebra $\mathcal{A}_\Sigma := \mathcal{A}(X^{\Omega_1}_{\mathbf{e}_0}) < h_{\mathbf{e}_0}, \pi / h^{q+1}_{\mathbf{e}_0} >= \mathcal{A}(X^{\Omega_1}_{\mathbf{e}_0}) < h_{\mathbf{e}_0} >$. Let $X^\Sigma_{\mathbf{e}_0} := sp(\mathcal{A}_\Sigma)$ be the affinoid space belonging to the affinoid algebra $\mathcal{A}_\Sigma =: \mathcal{A}(X^\Sigma_{\mathbf{e}_0})$. The affinoid space $X^\Sigma_{\mathbf{e}_0}$ is defined over the field L.

Using the function $h^\vee_{\mathbf{e}_0}$ instead of $h_{\mathbf{e}_0}$, one can define the affinoid algebra $\mathcal{A}^\vee_\Sigma := \mathcal{A}(X^{\Omega_1}_{\mathbf{e}_0}) < h^\vee_{\mathbf{e}_0} >$. The affinoid spaces $X^\Sigma_{\mathbf{e}_0} = sp(\mathcal{A}_\Sigma)$ and $sp(\mathcal{A}^\vee_\Sigma)$ define the same covering of $X^{\Omega_1}_{\mathbf{e}_0}$, but the covering is defined differently over the field L. They become identical over the field L with a $q+1$-th root of π added.

Let $\pi^{\frac{1}{q+1}}$ denote a $q+1$-th root of π. We can extend the action of the element $g_{\mathbf{e}_0} \in GU(2, L)$ from $\mathcal{A}(X^{\Omega_1}_{\mathbf{e}_0} \otimes L[\pi^{\frac{1}{q+1}}])$ to $\mathcal{A}(X^\Sigma_{\mathbf{e}_0} \otimes L[\pi^{\frac{1}{q+1}}])$. Indeed, the action of the element $g_{\mathbf{e}_0} \in GU(2, L)$ that permutes the two vertices in $\mathbf{e}_0$ can be defined by $g^*_{\mathbf{e}_0} h_{\mathbf{e}_0} = h^\vee_{\mathbf{e}_0} := \zeta \cdot \pi^{\frac{1}{q+1}} / h_{\mathbf{e}_0}$. Here $\zeta \in L$ is a unit root such that $\zeta^{q+1} = -1$. Note, that $\mathcal{A}_\Sigma \otimes L[\pi^{\frac{1}{q+1}}] = \mathcal{A}(X^\Sigma_{\mathbf{e}_0} \otimes L[\pi^{\frac{1}{q+1}}]) = \mathcal{A}(X^{\Omega_1}_{\mathbf{e}_0} \otimes L[\pi^{\frac{1}{q+1}}]) < h_{\mathbf{e}_0} >= \mathcal{A}(X^{\Omega_1}_{\mathbf{e}_0} \otimes L[\pi^{\frac{1}{q+1}}]) < h^\vee_{\mathbf{e}_0} >= \mathcal{A}^\vee_\Sigma \otimes L[\pi^{\frac{1}{q+1}}]$.

4.5 Lemma. *The map $\varphi_{\mathbf{e}_0} : X^\Sigma_{\mathbf{e}_0} \to X^{\Omega_1}_{\mathbf{e}_0}$ induced by the inclusion $\mathcal{A}(X^{\Omega_1}_{\mathbf{e}_0}) \subset \mathcal{A}(X^\Sigma_{\mathbf{e}_0})$ has degree $q+1$ and is étale.*

Proof. The degree of the map $X^{\Sigma}_{\mathbf{e}_0} \to X^{\Omega_1}_{\mathbf{e}_0}$ is clear from the definition. Let us look at the points of ramification. For convenience we will work over the field extension $L[\pi^{\frac{1}{q+1}}]$. The function $f_{\mathbf{e}_0}$ has absolute value 1 outside the affinoid subspaces $X^{\Omega_1}_{\mathbf{v}_0} \otimes L[\pi^{\frac{1}{q+1}}]$ and $X^{\Omega_1}_{\mathbf{v}_1} \otimes L[\pi^{\frac{1}{q+1}}]$ of $X^{\Omega_1}_{\mathbf{e}_0} \otimes L[\pi^{\frac{1}{q+1}}]$. Moreover, the element $g_{\mathbf{e}_0} \in GU(2, L)$, that permutes the vertices $\mathbf{v}_0$ and $\mathbf{v}_1$, also permutes the ramification points of the map $X^{\Sigma}_{\mathbf{e}_0} \otimes L[\pi^{\frac{1}{q+1}}] \to X^{\Omega_1}_{\mathbf{e}_0} \otimes L[\pi^{\frac{1}{q+1}}]$. Hence it is sufficient to look at the ramification points of $X^{\Sigma}_{\mathbf{v}_0} \otimes L[\pi^{\frac{1}{q+1}}] \to X^{\Omega_1}_{\mathbf{v}_0} \otimes L[\pi^{\frac{1}{q+1}}]$. Clearly, $f_{\mathbf{e}_0}(x) = 0$ can only occur for $x \in X^{\Omega_1}_{\mathbf{v}_0} \otimes L[\pi^{\frac{1}{q+1}}]$, if $1 + (\frac{x_1}{x_2})^{q-1} = 0$. Solving this equation over the residue field, gives the isotropic points $a \in \ell^*$. Since the ℓ-valued isotropic points do not occur in the reduction of $X^{\Omega_1}_{\mathbf{v}_0} \otimes L[\pi^{\frac{1}{q+1}}]$, the map is étale. This proves the lemma. □

4.6 Lemma. *Let $P_{\mathbf{e}_0} \subset SU(2, L)$ be the stabiliser of the edge $\mathbf{e}_0 \in \mathbf{b}$ and let $g \in P_{\mathbf{e}_0}$. Then the following holds:*

- **i)** *There exists a function $C_{0,g}(x) \in \mathcal{A}(X^{\Omega_1}_{\mathbf{e}_0})$ such that $g^* f_{\mathbf{e}_0}(x) = (\frac{x_2}{g^* x_2})^{q+1} \cdot C_{0,g}(x) \cdot f_{\mathbf{e}_0}(x)$. Moreover, $C_{0,g}(x) \equiv 1$ mod π if $1 \geq |\frac{x_1}{x_2}| > |\pi|$.*

- **ii)** *There exists a function $C_{1,g}(x) \in \mathcal{A}(X^{\Omega_1}_{\mathbf{e}_0})$ such that $g^* f_{\mathbf{e}_0}(x) = (\frac{g^* x_1}{x_1})^{q+1} \cdot C_{1,g}(x) \cdot f_{\mathbf{e}_0}(x)$. Moreover, $C_{1,g}(x) \equiv 1$ mod π if $1 > |\frac{x_1}{x_2}| \geq |\pi|$.*

Proof. We can write $f_{\mathbf{e}_0}(x) = \frac{1}{x_2^{q+1}} \cdot \frac{x_1^q x_2 + x_1 x_2^q}{1+(-\frac{\pi \cdot x_2}{x_1})^{(q-1)}}$. Furthermore, $C_{0,g}(x) = \frac{g^* x_2^{q+1} g^* f_{\mathbf{e}_0}(x)}{x_2^{q+1} f_{\mathbf{e}_0}(x)}$ satisfies the first part of statement (i) of the lemma. It remains to show that $C_{0,g}(x) \equiv 1$ mod π for $1 \geq \frac{x_1}{x_2} > |\pi|$.

Since $|\frac{\pi x_2}{x_1}| < 1$, $C_{0,g}(x) \equiv \frac{g^* x_1 g^* x_2^q + g^* x_1^q g^* x_2}{x_1^q x_2 + x_1 x_2^q}$ mod π holds. Any element $g \in P_{\mathbf{e}_0} \subset SU(2, L)$ preserves the hermitian form $h_2(x, y) = x_1 \overline{y_2} + x_2 \overline{y_1}$, therefore $g^* x_1 g^* x_2^q + g^* x_1^q g^* x_2 \equiv x_1^q x_2 + x_1 x_2^q$ mod π for $|\frac{x_1}{x_2}| = 1$. Moreover, $\frac{g^* x_1}{x_1} \equiv \frac{g^* x_2}{x_2} \equiv 1$ mod π if $1 > |\frac{x_1}{x_2}| > |\pi|$. Hence $C_{0,g}(x) \equiv \frac{g^* x_1 g^* x_2^q + g^* x_1^q g^* x_2}{x_1^q x_2 + x_1 x_2^q} \equiv 1$ mod π for $1 \geq |\frac{x_1}{x_2}| > |\pi|$. This proves statement (i) of the lemma.

The proof of statement (ii) of the lemma is similar. One uses the equality $f_{\mathbf{e}_0}(x) = -\frac{x_1^{q+1}}{\pi} \cdot \frac{1+(\frac{x_1}{x_2})^{(q-1)}}{x_1^q(-\pi x_2) + x_1(-\pi x_2)^q}$ and the fact that the reduction of the hermitian form $g^*_{\mathbf{e}_0} h_2$ at the vertex $\mathbf{v}'$ equals $x_1^q(-\pi x_2) + x_1(-\pi x_2)^q$ modulo π.

□

4.7. The group action. Using the lemma above one can define the action of $P_{\mathbf{e}_0}$ on $\mathcal{A}(X^{\Sigma}_{\mathbf{e}_0})$. There exist functions $c_{0,g}(x)$ and $c_{1,g}(x)$ well-defined on the open admissable subsets $1 \geq |\frac{x_1}{x_2}| > |\pi|$ and $1 > |\frac{x_1}{x_2}| \geq |\pi|$ of $X^{\Omega_1}_{\mathbf{e}_0}$, respectively, such that $c_{i,g}(x)^{q+1} = C_{i,g}(x)$ and $c_{i,g}(x) \equiv 1 \bmod \pi$ for $i = 1, 2$. Then the action of $g \in P_{\mathbf{e}_0}$ on $h_{\mathbf{e}_0}$ is defined as follows:

$$g^* h_{\mathbf{e}_0}(x) = \tfrac{x_2}{g^* x_2} \cdot c_{0,g}(x) \cdot h_{\mathbf{e}_0}(x) \text{ if } 1 \geq |\tfrac{x_1}{x_2}| > |\pi|.$$

$$g^* h_{\mathbf{e}_0}(x) = \tfrac{g^* x_1}{x_1} \cdot c_{1,g}(x) \cdot h_{\mathbf{e}_0}(x) \text{ if } 1 > |\tfrac{x_1}{x_2}| \geq |\pi|.$$

Since $(\frac{x_2}{g^* x_2})^{q+1} \cdot C_{0,g}(x) = (\frac{g^* x_1}{x_1})^{q+1} \cdot C_{1,g}(x)$ and $\frac{g^* x_1}{x_1} \equiv \frac{g^* x_2}{x_2} \equiv 1 \bmod \pi$ for $1 > |\frac{x_1}{x_2}| > |\pi|$, these actions coincide when they are both defined. Since $g_1^*(g_2^* f_{\mathbf{e}_0}(x)) = (g_1 g_2)^* f_{\mathbf{e}_0}(x))$, it follows from the definition that $g_1^*(g_2^* h_{\mathbf{e}_0}(x)) = (g_1 g_2)^* h_{\mathbf{e}_0}(x))$.

Moreover, the first formula also defines the action of $g \in P_{\mathbf{v}_0}$ on $\mathcal{A}(X^{\Sigma}_{\mathbf{v}_0})$, since statement (i) of the lemma above still holds in this case. Similarly, the second formula defines the action of $g \in P_{\mathbf{v}_1}$ on $\mathcal{A}(X^{\Sigma}_{\mathbf{v}_1})$.

4.8 Theorem. **(a)** *The affinoid spaces $X^{\Sigma}_{\mathbf{e}}$, $X^{\Sigma}_{\mathbf{v}}$ for vertices $\mathbf{v} \in \mathbf{b}$ and edges $\mathbf{e} \in \mathbf{b}$ glue together and form a pure affinoid covering of a separated analytical space Σ.*

(b) *The map $\varphi : \Sigma \to \Omega_1$ obtained by glueing the maps $\varphi_{\mathbf{e}}$ is $SU(2, L)$-equivariant and has degree $q + 1$.*

(c) *The map $\varphi : \Sigma \to \Omega_1$ is étale.*

(d) *Then the reduction of Σ is as follows:*

1. *For each vertex of the building $\mathbf{b}$ curves isomorphic to the plane hermitian projective curve given by $x_0^{q+1} + x_1 x_2^q + x_1^q x_2 = 0$ in $\mathbb{P}^2_{\ell}$. This curve is non-singular.*
2. *The components belonging to two vertices $\mathbf{v}_1$ and $\mathbf{v}_2$ of the building $\mathbf{b}$ intersect in a ℓ-valued point if and only if the two vertices form an edge $\mathbf{e}$ of the building.*

Proof. That the affinoids $X^{\Sigma}_{\mathbf{e}}$ for edges $\mathbf{e} \in \mathbf{b}$ glue together and form some rigid analytic space Σ is clear. The only point of concern is the separatedness of the resulting analytical space Σ. However, from the fact that we have maps

$\varphi_{\mathbf{e}}: X^{\Sigma}_{\mathbf{e}} \to X^{\Omega_1}_{\mathbf{e}}$ that coincide on $X^{\Sigma}_{\mathbf{v}}$ for all edges $\mathbf{e} \ni \mathbf{v}$ the separatedness follows. This proves statement (a).

Statements (b) and (c) of the theorem are clear from the construction of the maps $\varphi^{\Sigma}_{\mathbf{e}}: X^{\Sigma}_{\mathbf{e}} \to X^{\Omega_1}_{\mathbf{e}}$ and the lemma above.

So let us now prove statement (d) of the theorem. First we consider the reduction of the affinoid $X^{\Sigma}_{\mathbf{e}}$. We only have to consider the component corresponding to the vertex $\mathbf{v}$. The other component is isomorphic to it, since the element $g_{\mathbf{e}} \in GU(2, L)$ defined above permutes the vertices in $\mathbf{e}$ and preserves $\mathcal{A}(X^{\Sigma}_{\mathbf{e}} \otimes L[\pi^{\frac{1}{q+1}}])$. The generators of the affine ℓ-algebra giving the component for the vertex $\mathbf{v}$ are $\overline{h_{\mathbf{e}}(x)}$ and $\overline{\frac{x_1}{x_2}}$ and satisfy the equation $\overline{h_{\mathbf{e}}(x)}^{q+1} = -\overline{\frac{x_1}{x_2}} - \overline{\frac{x_1}{x_2}}^q$. Furthermore, $\overline{\frac{x_1}{x_2}} \neq a$ for isotropic $a \in \ell - \{0\}$, since the ℓ-valued isotropic points are omitted.

Let us now compare this affine algebra with an open affine subset of the curve $\mathcal{C} \subset \mathbb{P}^2_{\ell}$ given by the equation $x_0^{q+1} + x_1 x_2^q + x_1^q x_2 = 0$. Taking $x_2 \neq 0$, we obtain the equation $\frac{x_0}{x_2}^{q+1} + \frac{x_1}{x_2} + \frac{x_1}{x_2}^q = 0$. Removing the isotropic points $\frac{x_0}{x_2} \neq a$ for $a \in \ell - \{0\}$ results in an affine subset $A \subset \mathcal{C}$ isomorphic to the component of the reduction of the affinoid space $X^{\Sigma}_{\mathbf{e}}$ belonging to the vertex $\mathbf{v} \ni \mathbf{e}$. The group $SU(2, \ell)$ acts on $\mathbb{P}^2_{\ell}$ fixing the point x_0. This action of $SU(2, \ell)$ preserves the curve $\mathcal{C} \subset \mathbb{P}^2_{\ell}$. The affine sets $g(A)$ for $g \in SU(2, \ell)$ cover the curve $\mathcal{C}$.

Let $P_{\mathbf{v}} \subset SU(2, L)$ denote the stabilizer of the vertex $\mathbf{v} \in \mathbf{b}$. Then the component of the reduction of $X^{\Sigma}_{g(\mathbf{e})} \otimes L[\pi^{\frac{1}{q+1}}]$ belonging to the vertex $\mathbf{v}$ corresponds to the affine space $\overline{g}(A)$, where $\overline{g} \in SU(2, \ell)$ denotes the reduction of $g \in P_{\mathbf{v}}$. Hence the component of the reduction of Σ belonging to the vertex $\mathbf{v}$ is indeed a curve $\mathcal{C}$ as stated in the theorem.

A calculation of the partial derivatives shows that the curve $\mathcal{C}$ is non-singular. □

4.9. Embedding the affinoid $X^{\Sigma}_{\mathbf{e}}$ into a projective plane. Let us embed the affinoid space $X^{\Sigma}_{\mathbf{e}}$ into $\mathbb{P}^2_L$. Let $x_i,\ i = 0, 1, 2$ be the coordinates of $\mathbb{P}^2_L$ and let $x_0 = 0$ define a projective line $\mathbb{P}^1_L$ inside the $\mathbb{P}^2_L$. We take the usual unitary form $x_1\overline{x_2} + x_2\overline{x_1}$ on the projective line $\mathbb{P}^1_L$. It is preserved by a group $SU(2, L)$ acting on the projective line. Then $\Omega_1 \subset \mathbb{P}^1_L$ is obtained by removing the L-valued isotropic points. The affinoid $X^{\Omega_1}_{\mathbf{e}_0} \subset \Omega_1$ consists of the points $(0, x_1, x_2)$ such that $1 \geq |\frac{x_1}{x_2}| \geq |\pi|$ with the non-zero ℓ-valued isotropic points removed from the reduction of the affinoid space. An embedding of the affinoid space $X^{\Sigma}_{\mathbf{e}}$ can now be obtained by taking the points $x \in \mathbb{P}^2_L$ such that

$(0,x_1,x_2)\in X^{\Omega_1}_{\mathbf{e}_0}$ and, moreover, $(\frac{x_0}{x_2})^{q+1}=-f_{\mathbf{e}_0}((x_1,x_2))=-\frac{x_1}{x_2}\cdot\frac{1+(\frac{x_1}{x_2})^{q-1}}{1+(-\frac{\pi\cdot x_2}{x_1})^{q-1}}$. Therefore $\frac{x_0}{x_2}$ represents the function $h_{\mathbf{e}_0}((x_1,x_2))$.

Let us use a second projective plane $\mathbb{P}^2_L$ with coordinates $z_i,\ i=1,2,3$ to obtain the alternative embedding $X^{\Sigma}_{\mathbf{e}}$ into $\mathbb{P}^2_L$ based on the function $h^{\vee}_{\mathbf{e}_0}$. We relate the two projective planes using the relation $x_1z_2+x_2z_1=0$. This identifies the two projective lines $\mathbb{P}^1_L$ given by $x_0=0$ and by $z_0=0$, respectively. The action of the group $SU(2,L)$ on the $\mathbb{P}^1_L$ using the coordinates x_1,x_2 and z_1,z_2 differs by the action of a generator of the Galois group $Gal(L/K)$. Now we can describe the embedding of the affinoid space $X^{\Sigma}_{\mathbf{e}}$ using the coordinates $z_i,\ i=0,1,2$. A straightforward approach would be to use equation $(\frac{z_0}{z_2})^{q+1}=\pi/f_{\mathbf{e}_0}((z_1,z_2))$ to define the embedding of the affinoid space $X^{\Sigma}_{\mathbf{e}}$, but then the action of the element $g_{\mathbf{e}_0}$ would not be defined over L. To avoid this, we use the equation $\pi\cdot(\frac{z_0}{z_2})^{q+1}=\pi/f_{\mathbf{e}_0}((z_1,z_2))$ instead. Then $X^{\Sigma}_{\mathbf{e}}$ consists of the points $z\in\mathbb{P}^2_L$ such that $(0,z_1,z_2)\in X^{\Omega_1}_{\mathbf{e}_0}$ and $(\frac{z_0}{z_2})^{q+1}=1/f_{\mathbf{e}_0}((z_1,z_2))$.

We can now define an action of the element $g_{\mathbf{e}_0}\in GU(2,L)$ that permutes the two projective planes as follows: $g_{\mathbf{e}_0}(x)=z=(\frac{x_1x_2}{x_0},-x_2,\pi x_1)$ and $g_{\mathbf{e}_0}(z)=x=(\frac{z_1z_2}{z_0},-z_2,\pi z_1)$. This is well-defined, once the coordinate lines $x_i=0$ and $z_i=0,\ i=0,1,2$ are removed from the two projective planes. Then $g^2_{\mathbf{e}_0}$ acts as $g^2_{\mathbf{e}_0}=-\pi\cdot id.$ on both projective planes (with the coordinate lines removed). For a point $x\in X^{\Sigma}_{\mathbf{e}}$, the image $z=g_{\mathbf{e}_0}(x)$ satisfies: $(\frac{z_0}{z_2})^{q+1}=(\frac{x_1x_2}{x_0}/(\pi\cdot x_1))^{q+1}=(\frac{x_2}{\pi\cdot x_0})^{q+1}=(\frac{-f_{\mathbf{e}_0}(x)}{\pi})^{q+1}=(\frac{1}{f_{\mathbf{e}_0}(g_{\mathbf{e}_0}(x))})^{q+1}=(\frac{1}{f_{\mathbf{e}_0}((z_1,z_2))})^{q+1}$. Hence $g_{\mathbf{e}_0}(x)$ is contained in the other embedding of the affinoid $X^{\Sigma}_{\mathbf{e}}$. Therefore both embeddings of the affinoid space $X^{\Sigma}_{\mathbf{e}}$ are permuted by the element $g_{\mathbf{e}_0}\in GU(2,L)$.

Since the affinoid $X^{\Sigma}_{\mathbf{e}}$ can be embedded into the projective plane and the components of the reduction of Σ consist of plane projective curves, one can embed the complete affinoid covering of Σ in the plane. However, in general such an embedding does not have a group action on it that extends to the entire projective plane.

4.10. Open admissable subspaces. For later use some open admissable subspaces of Σ are defined. For a vertex $\mathbf{v}\in\mathbf{b}$ we define the subspace $\Sigma_{\mathbf{v}}:=\{x\in\Sigma\mid d_{\mathbf{b}}(\psi(\varphi(x)),\mathbf{v})<1\}$. For an edge $\mathbf{e}\in\mathbf{b}$ we take $\Sigma_{\mathbf{e}}:=\bigcap_{\mathbf{v}\in\mathbf{e}}\Sigma_{\mathbf{v}}=\{x\in\Sigma\mid\psi(\varphi(x))\in\mathbf{e},\ \psi(\varphi(x))\neq\mathbf{v}\text{ for vertices }\mathbf{v}\in\mathbf{e}\}$. The covering $\{\Sigma_{\mathbf{v}},\Sigma_{\mathbf{e}}\mid\mathbf{v},\mathbf{e}\in\mathbf{b}\}$ is an open admissable covering of Σ.

The group action on the affinoid $X^{\Sigma}_{\mathbf{e}}$ is defined by giving an explicit group action on $X_{\mathbf{e}}^{\Sigma} \bigcap \Sigma_{\mathbf{v}}$, where $\mathbf{v} \in \mathbf{e}$ is the vertex of type $\tau(\mathbf{v}) = 0$ and on $X_{\mathbf{e}}^{\Sigma} \bigcap \Sigma_{\mathbf{v}'}$ where $\mathbf{v}' \in \mathbf{e}$ is the vertex of type $\tau(\mathbf{v}') = 1$ that coincides on the intersection $\Sigma_{\mathbf{e}} = X_{\mathbf{e}}^{\Sigma} \bigcap \Sigma_{\mathbf{v}} \bigcap \Sigma_{\mathbf{v}'}$. In sections §6 and §7 we will embed $\bigcup_{\mathbf{v} \in \mathbf{b}, \tau(\mathbf{v})=0} \Sigma_{\mathbf{v}}$ and $\bigcup_{\mathbf{v} \in \mathbf{b}, \tau(\mathbf{v})=1} \Sigma_{\mathbf{v}}$ into projective planes such that a discrete subgroup of the group $SU(2, L)$ acts linearly on it.

5 Definition of the étale covering over the field K

We define a rigid analytic variety $\Sigma^{(q-1)}$ over the field K, such that $\Sigma^{(q-1)} \otimes L$ consists of $q-1$ connected components isomorphic to the space Σ. An $SL(2, K)$-equivariant étale map $\Sigma^{(q-1)} \to \Omega_{1,K} := \mathbb{P}^1_K - \mathbb{P}^1(K)$ is defined. The étale map factors through a space ${\Omega_1}^{(q-1)}$ defined over the field K. Here the space ${\Omega_1}^{(q-1)} \otimes L$ consists of $q-1$ connected components all isomorphic to the p-adic upper half plane Ω_1.

We compare the analytic space $\Sigma^{(q-1)}$ with the description by Teitelbaum (See [Te]) of the first level of a system of étale coverings of $\Omega_{1,K}$ defined by Drinfel'd. The definition of the space $\Sigma^{(q-1)}$ over the field K differs from the one that Drinfel'd uses. The spaces are isomorphic over a suitably choosen finite unramified extension of K. Drinfel'd defines the étale coverings over the maximal unramified extension $K^{nr} \supset K$ and uses a Frobenius map to imply a definition of the spaces over the field K. Our construction therefore uses a different Frobenius map on $\Sigma^{(q-1)} \otimes K^{nr}$.

5.1. Multiples of the p-adic upper halfplane defined over the field K. The space Ω_1 itself is defined over the field L, therefore even multiples can be defined over the field K in such a way that the Galois group $Gal(L/K)$ acts by permutation of pairs of connected components. We describe them explicitly embedded in a projective space $\mathbb{P}^3_K$ in such a way that the group $SU(2, L)$ acts naturally on them and such that the action is defined over K.

To obtain our embeddings of multiples of Ω_1, we consider a vector space $V_L =< e_1, e_2 >\cong L^2$ with a unitary form $h_2(x,x) = x_1\bar{x_2} + x_2\bar{x_1}$ on it as a K-space $V_K \cong K^4$. Let $x_1 := y_1 + \omega y_2$ and $x_2 := y_3 + \omega y_4$ with $\overline{\omega} = -\omega$. We assume that $\omega \in L$ is a root of unity in L. Then $\overline{\omega} = \omega^q$ and $\omega^{q-1} = -1$. In $\mathbb{P}^3_K = \mathbb{P}(V_K)$ we consider the variety X_{ω^2} that is the image of the vectors $x \in V_L$, such that both $h_2(x,x) = x_1\bar{x_2} + x_2\bar{x_1} = 0$ and $h_2(g_\omega(x), g_\omega(x)) = x_1\frac{\bar{x_2}}{\omega} + \frac{x_2}{\omega}\bar{x_1} = \frac{1}{\omega}(-x_1\bar{x_2} + x_2\bar{x_1}) = 0$. Her g_ω is the diagonal element $g_\omega := diag(1, \omega^{-1})$ w.r.t. the basis e_1, e_2 of V_L. The group $SU(2, L)$ preserves the variety X_{ω^2}. In the coordinates y_i we obtain the relations $y_1 y_3 - \omega^2 y_2 y_4 = 0$ $(*)$ and $y_2 y_3 - y_1 y_4 = 0$ $(**)$.

The variety X_{ω^2} is defined over K and $X_{\omega^2} \otimes L$ consists of two copies of the projective line $\mathbb{P}^1_L$. The variety X_{ω^2} consists of the points $(y_1, y_2, y_3, y_4) = (\epsilon\omega y_2, y_2, \epsilon\omega y_4, y_4)$ with $\epsilon = \pm 1$. The p-adic upper halfplane Ω_1 in each of these components $\mathbb{P}^1_L$ is obtained by omitting the points y such that $(y_1, y_4) \in \mathbb{P}^1(K)$. Let ${\Omega_1}^{(2)}$ be the rigid analytic variety ${\Omega_1}^{(2)} := X_{\omega^2} - \{y \in X_{\omega^2} \mid (y_1, y_4) \in \mathbb{P}^1(K)\}$. Then ${\Omega_1}^{(2)}$ is defined over K and ${\Omega_1}^{(2)} \otimes L$ consists of two copies of Ω_1.

Different choices of the unit root $\omega \in L$, $\omega^{q-1} = -1$, give different varieties X_{ω^2}. Since $(\omega/\omega')^{q-1} = 1$ for $\omega, \omega' \in L$ such that $\omega^{q-1} = (\omega')^{q-1} = -1$, the quotient ω/ω' is a $q-1$-th root of unity and contained in K. There exists $(q-1)/2$ distinct varieties $X_{\omega^2} \subset \mathbb{P}^3_K$. Let $X_\dagger$ be the union $\bigcup\{X_{\omega^2} \mid \omega \in L^\circ, \omega^q = -\omega\}$. The variety $X_\dagger$ can be defined by replacing the equation $(*)$ by the equation $(y_1 y_3)^{\frac{q-1}{2}} = -(y_2 y_4)^{\frac{q-1}{2}}$ $(***)$. The variety $X_\dagger \otimes L$ consists of $q-1$ copies of $\mathbb{P}^1_L$. We define ${\Omega_1}^{(q-1)}$ as ${\Omega_1}^{(q-1)} := X_\dagger - \{y \in X_\dagger \mid (y_1, y_4) \in \mathbb{P}^1(K)\}$. Then ${\Omega_1}^{(q-1)}$ is a rigid analytic variety defined over K and ${\Omega_1}^{(q-1)} \otimes L$ consists of $q-1$ connected components all isomorphic to Ω_1.

5.2. Group action on ${\Omega_1}^{(q-1)}$. Let μ_{q-1} be the group $\mu_{q-1} := \{c \in K^\circ \mid c^{q-1} = 1\}$ consisting of the $q-1$-th roots of unity in K° and let $\mu_2 \subset \mu_{q-1}$ be the subgroup $\mu_2 = \{\pm 1\}$. We define an action of the group μ_{q-1} on the variety $X_\dagger$ that is diagonal with respect to the coordinates y_i, $i = 1, \ldots, 4$. Let $c \cdot y = (cy_1, y_2, y_3, c^{-1} y_4)$ for $y \in X_\dagger$. The cyclic group μ_{q-1} permutes the connected components of $X_\dagger \otimes L$ transitively. The subgroup $\mu_2 \subset \mu_{q-1}$ fixes each of the subvarieties $X_{\omega^2} \subset X_\dagger$ and permutes the two connected components of X_{ω^2}. The action of the group μ_2 differs from the action of the Galois group $Gal(L/K)$ by an action of the diagonal element $diag(1, -1)$. The actions coincide when the point $y \in X_{\omega^2}$ is such that $(y_1, y_4) \in \mathbb{P}^1_K$. Therefore the action differs on the variety ${\Omega_1}^{(2)}$ and the quotient map ${\Omega_1}^{(2)} \to {\Omega_1}^{(2)}/\mu_2 \cong \Omega_{1,K}$ is étale of degree two and defined over the field K.

To define a group action on the variety ${\Omega_1}^{(q-1)}$ we assume that the point $y = (y_1, y_2, y_3, y_4) \in X_\dagger$ such that $y_1 = c\omega y_2$ and $y_3 = c\omega y_4$ is the point $\sum_{i=1}^4 y_i f_i$. Here the basis f_i, $i = 1, \ldots, 4$ of V_K is defined by $f_1 = e_1$, $f_2 = c^{-1}\omega^{-1} e_1$, $f_3 = c e_2$ and $f_4 = \omega^{-1} e_2$. The usual group action of the group $SU(2, L)$ w.r.t. the vectors e_i, $i = 1, 2$, results in a group action of a group $SL(2, K)$ w.r.t. the vectors f_1 and f_4 and w.r.t. the vectors f_2 and f_3.

The action of the element

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SU(2, L)$$

w.r.t. the basis e_1, e_2, becomes the action of the elements

$$\begin{pmatrix} a_{11} & a_{12}\omega \\ a_{21}/\omega & a_{22} \end{pmatrix} \text{ and } \begin{pmatrix} a_{11} & a_{12}/c^2\omega \\ a_{21}c^2\omega & a_{22} \end{pmatrix} \in SL(2, K)$$

w.r.t. the vectors f_1, f_4 and the vectors f_2, f_3, respectively. Therefore the quotient map ${\Omega_1}^{(q-1)} \to {\Omega_1}^{(q-1)}/\mu_{q-1} \cong \Omega_{1,K}$ given by $y = (y_1, y_2, y_3, y_4) \to (cy_1, c^{-1}y_4) \sim (c^2y_1, y_4)$ for $y \in {\Omega_1}^{(q-1)} \bigcap X_{c^2\omega^2} \subset X_\dagger$ is $SL(2, K)$-equivariant. The map is étale of degree $q-1$ and defined over K. (Alternatively, one could define the map ${\Omega_1}^{(q-1)} \to \Omega_{1,K}$ by $y = (y_1, y_2, y_3, y_4) \to (y_2, y_3)$. Then the action of the group $SL(2, K)$ differs on the subspaces ${\Omega_1}^{(q-1)} \bigcap X_{c^2\omega^2}$ by conjugation with the diagonal element $diag(c^{-1}, c)$ w.r.t. the vectors f_2, f_3.)

The action of the element $g_{\mathbf{e}_0} \in GU(2, L)$ on $y \in {\Omega_1}^{(q-1)} \bigcap X_{c^2\omega^2}$ is given by $g_{\mathbf{e}_0}((y_1, y_2, y_3, y_4)) = (\frac{\pi}{c}y_3, \frac{\pi}{c}y_4, -cy_1, -cy_2) \sim (\pi y_3, \pi y_4, -c^2y_1, -c^2y_2)$. This action preserves the connected components of ${\Omega_1}^{(q-1)}$. One can twist this action by the element $-1 \in \mu_{q-1}$ and obtain an action that interchanges the components. Let $g^\dagger_{\mathbf{e}_0}$ denote the element $g^\dagger_{\mathbf{e}_0} := g_{\mathbf{e}_0}h_{-1} = h_{-1}g_{\mathbf{e}_0}$, where $h_{-1} \in \mu_{q-1}$ is the element representing -1, i.e $h_{-1}y = (-y_1, y_2, y_3, -y_4)$. Then $g^\dagger_{\mathbf{e}_0}(y) = (-\pi y_3, \pi y_4, -c^2y_1, c^2y_2)$. The element $g^\dagger_{\mathbf{e}_0}$ permutes the two connected components of ${\Omega_1}^{(q-1)} \bigcap X_{c^2\omega^2}$ for each value of the $q-1$-th root of unity $c \in K^\circ$.

In the proposition below we compile the properties of ${\Omega_1}^{(2)}$ and ${\Omega_1}^{(q-1)}$ that have been established:

5.3 Proposition. *The following statements hold:*

i) There exists a rigid analytic variety ${\Omega_1}^{(2)}$ defined over the field K such that the following statements hold:

a) ${\Omega_1}^{(2)} \otimes L$ consists of two connected components that are both isomorphic to Ω_1.

b) $Gal(L/K)$ permutes the two connected components of ${\Omega_1}^{(2)} \otimes L$.

c) The quotient map ${\Omega_1}^{(2)} \to {\Omega_1}^{(2)}/\mu_2 \cong \Omega_{1,K}$ is an $SL(2, K)$-equivariant finite étale map of degree 2 defined over the field K.

ii) There exists a rigid analytic variety ${\Omega_1}^{(q-1)}$ defined over the field K such that the following statements hold:

a) ${\Omega_1}^{(q-1)} \otimes L$ consists of $q-1$ connected components that are all isomorphic to Ω_1.

b) There is an action of the cyclic group $\mu_{q-1} \cong \mathbb{F}_q^*$ on ${\Omega_1}^{(q-1)}$ that permutes the connected components of ${\Omega_1}^{(q-1)} \otimes L$ transitively.

c) The quotient map ${\Omega_1}^{(q-1)} \to {\Omega_1}^{(q-1)}/\mu_{q-1} \cong \Omega_{1,K}$ is an $SL(2,K)$-equivariant finite étale map of degree $q-1$ defined over the field K.

5.4. Example. We give an example to indicate how a discrete group acts on the analytic space ${\Omega_1}^{(2)}$. Let $\Delta \subset GU(2,L) \cong GL(2,K)$ be a discrete co-compact subgroup. Let $\Delta_0 \subset \Delta$ be the subgroup $\Delta_0 := \{\gamma \in \Delta \mid v(det(\gamma)) = 0\}$ and let $\Delta_2 \subset \Delta$ be the subgroup $\Delta_2 := \{\gamma \in \Delta \mid v(det(\gamma)) \in 2\mathbb{Z}\}$. We assume that $\Delta_0 = \Delta \bigcap SU(2,L)$ holds. We assume that the groups Δ and Δ_2 contain an element $\gamma_2 = \pi \cdot Id. = diag(\pi,\pi)$ for some uniformising element $\pi \in L^\circ$ with $v(\pi) = 1$. The subgroup $\Delta_2 \subset \Delta$ is generated by Δ_0 and the element γ_2. We also assume that the group Δ contains an element γ_1 with $v(det(\gamma_1)) = 1$. Then Δ/Δ_2 is a group of order two that acts on the vertices $\mathbf{v}$ of the quotient $\mathbf{b}/\Delta_2 = \mathbf{b}/\Delta_0$ by interchanging the vertices of type $\tau(\mathbf{v}) = 0$ with the vertices of type $\tau(\mathbf{v}) = 1$. By m_Δ we denote the number of vertices $\mathbf{v}$ in the quotient $\mathbf{b}/\Delta_0 = \mathbf{b}/\Delta_2$.

The group $\Delta/ < \gamma_2 >$ acts discretely on ${\Omega_1}^{(2)}$. We let the element $\gamma_1 \in \Delta$ act on ${\Omega_1}^{(2)}$ through the dagger action $\gamma_1^\dagger := \gamma_1 h_{-1}$. Let $\Delta^\dagger$ denote the group $\Delta^\dagger :=< \gamma_1^\dagger, \Delta_2 > / < \gamma_2 >$. The quotient ${\Omega_1}^{(2)}/\Delta^\dagger$ is defined over the field K and consists of a single connected component. The reduction of ${\Omega_1}^{(2)}/\Delta^\dagger$ consists of $m_\Delta/2$ components that are isomorphic to curves $X_{\omega^2} \otimes k$. The curve $({\Omega_1}^{(2)}/\Delta^\dagger) \otimes L$ has a reduction consisting of m_Δ curves $\mathbb{P}^1_\ell$. The group Δ/Δ_2 acts on the curve $({\Omega_1}^{(2)}/\Delta^\dagger) \otimes L$ as the Galois group $Gal(L/K)$.

Since the analytic variety ${\Omega_1}^{(2)}$ is not a connected space, a quotient of ${\Omega_1}^{(2)}$ by a group acting discretely on ${\Omega_1}^{(2)}$ does not constitute a uniformisation of the quotient curve. In particular, this holds for the curve ${\Omega_1}^{(2)}/\Delta^\dagger$, even though it consists of a single connected component. The curve ${\Omega_1}^{(2)}/\Delta^\dagger$ has a uniformisation defined over the field L by the simply connected variety Ω_1. Since $\Delta_2/ < \gamma_2 > \cong \Delta_0$ and $\Delta^\dagger/(\Delta_2/ < \gamma_2 >) \cong \mathbb{Z}/2\mathbb{Z}$, the element $\gamma_1^\dagger$ mod $\Delta_2/ < \gamma_2 >$ acts on the curve $({\Omega_1}^{(2)}/\Delta_0) \otimes L$ as an element of order two permuting the two connected components of the curve. Therefore the curves

$({\Omega_1}^{(2)}/\Delta^\dagger) \otimes L$ and Ω_1/Δ_0 are isomorphic. Let us for convenience assume that the group Δ_0 does not contain non-trivial elements of finite order. Then the uniformisation of the curve $({\Omega_1}^{(2)}/\Delta^\dagger) \otimes L$ is given by the description of this curve as the quotient of Ω_1 by the discrete group Δ_0. In particular, the curve ${\Omega_1}^{(2)}/\Delta^\dagger$ is a form of the curve Ω_1/Δ_0 defined over the field K.

5.5 Theorem. *The following statements hold:*

i) There exists a rigid analytic variety $\Sigma^{(2)}$ defined over the field K such that the following statements hold:

- *a) $\Sigma^{(2)} \otimes L$ consists of two connected components that are both isomorphic to Σ.*
- *b) Gal(L/K) permutes the two connected components of $\Sigma^{(2)} \otimes L$.*
- *c) The action of the element $g^\dagger_{\mathbf{e}_0}$ for $g_{\mathbf{e}_0} \in GU(2, L)$ permutes the two connected components of $\Sigma^{(2)} \otimes L$.*
- *d) There exists a finite étale map $\Sigma^{(2)} \to {\Omega_1}^{(2)}$ defined over the field K of degree $q + 1$.*
- *e) There exists a finite étale map $\Sigma^{(2)} \to {\Omega_1}^{(2)}/\mu_2 \cong \Omega_{1,K}$ defined over the field K.*

ii) There exists a rigid analytic variety $\Sigma^{(q-1)}$ defined over the field K such that the following statements hold:

- *a) $\Sigma^{(q-1)} \otimes L$ consists of $q - 1$ connected components that are all isomorphic to Σ.*
- *b) There is an action of the cyclic group $\mu_{q-1} \cong \mathbb{F}_q^*$ on $\Sigma^{(q-1)}$ that permutes the connected components of $\Sigma^{(q-1)} \otimes L$ transitively.*
- *c) The action of the element $g^\dagger_{\mathbf{e}_0}$ for $g_{\mathbf{e}_0} \in GU(2, L)$ permutes the connected components of each $\Sigma^{(2)} \otimes L$ contained in $\Sigma^{(q-1)} \otimes L$.*
- *d) There exists a finite étale map $\Sigma^{(q-1)} \to {\Omega_1}^{(q-1)}$ defined over the field K of degree $q + 1$.*
- *e) There exists a finite étale map $\Sigma^{(q-1)} \to {\Omega_1}^{(q-1)}/\mu_{q-1} \cong \Omega_{1,K}$ of degree $q^2 - 1$ defined over the field K.*
- *f) The Galois group of the covering $\Sigma^{(q-1)} \to \Omega_{1,K}$ is isomorphic to the group $(L^\circ/\pi L^\circ)^* \cong \mathbb{F}^*_{q^2}$.*

Proof. The spaces $\Sigma^{(2)}$ and $\Sigma^{(q-1)}$ can be obtained by glueing affinoids. Let us briefly sketch the construction of the affinoids $X_{\mathbf{e}}^{\Sigma^{(q-1)}}$, $\mathbf{e} \in \mathbf{b}$, for the case $\Sigma^{(q-1)}$. For the standard edge $\mathbf{e}_0 \in \mathbf{b}$ one takes the points $y \in {\Omega_1}^{(q-1)}$ such that $(y_2, y_4) \in X_{\mathbf{e}_0}^{\Omega_1} \subset \Omega_1$. The function $f_{\mathbf{e}_0}$ is replaced by the function $f_{\mathbf{e}_0,(q-1)}(y) := \frac{y_2}{y_4} \cdot \frac{1+(\frac{y_2}{y_4})^{q-1}}{1+(\frac{\pi y_4}{y_2})^{q-1}}$. The affinoid $X_{\mathbf{e}_0}^{\Sigma^{(q-1)}}$ is defined by adding a $q+1$-th root of the function $f_{\mathbf{e}_0,(q-1)}(y)$.

Then all statements of the theorem except ii(f) are straightforward combinations of properties of the multiples ${\Omega_1}^{(2)}$ and ${\Omega_1}^{(q-1)}$ of the p-adic upper halfplane collected in the proposition above and properties of the étale covering Σ of Ω_1.

Statement ii(f) is a direct consequence of Drinfel'ds construction. We will describe the Galois group and its action explicitly. We identify $(L^\circ/\pi L^\circ)^*$ with the group $\mu_{q^2-1} \subset L^\circ$ of q^2-1-th roots of unity. A q^2-1-th root of unity ζ acts on a point $y \in \Sigma^{(q-1)} \subset \mathbb{P}_K^4$ as follows:
$\zeta \cdot y = (\zeta y_0, N_{L/K}(\zeta) y_1, y_2, y_3, N_{L/K}(\zeta)^{-1} y_4)$. Here $N_{L/K} : L \to K$ denotes the norm map. The subgroup $\{\zeta \in \mu_{q^2-1} \mid N_{L/K}(\zeta) = 1\} \subset \mu_{q^2-1}$ preserves all the connected components of $\Sigma^{(q-1)}$. □

5.6. Hermitian curves defined over the field $\mathbb{F}_q$. Before we compare the variety $\Sigma^{(q-1)}$ with Drinfel'ds étale covering of $\Omega_{1,K}$, we consider curves whose connected components are hermitian curves and that can be defined over the finite field $\mathbb{F}_q$.

Let $\mathbb{P}^2_{\mathbb{F}_q}$ be a projective plane over $\mathbb{F}_q$ with coordinates y_i, $i = 0, 1, 2$. Let $\mathcal{H}_0 \subset \mathbb{P}^2_{\mathbb{F}_q}$ be the hermitian curve defined by the equation $y_0^{q+1} + y_1 y_2^q - y_2 y_1^q = 0$.

Let $\mathbb{P}^2_{\mathbb{F}_{q^2}}$ be the projective plane defined over $\mathbb{F}_{q^2}$ with coordinates x_i, $i = 0, 1, 2$. Let $\mathcal{H}_1 \subset \mathbb{P}^2_{\mathbb{F}_{q^2}}$ be the hermitian curve defined by the equation $x_0^{q+1} + x_1 x_2^q + x_2 x_1^q = 0$. We denote by $\mathcal{H}_1^{(2)}$ a curve consisting of two connected components that are both isomorphic to the hermitian curve $\mathcal{H}_1$.

Let $\widetilde{\omega} \in \mathbb{F}_{q^2}$ be such that $\widetilde{\omega}^{q-1} = -1$. Let $\mathcal{H}_2$ be the hermitian curve defined by the equation $\widetilde{\omega} x_0^{q+1} + x_1 x_2^q - x_2 x_1^q = 0$ in $\mathbb{P}^2_{\mathbb{F}_{q^2}}$. By $\mathcal{H}_2^{(2)}$ we denote a curve consisting of two connected components that are both isomorphic to the hermitian curve $\mathcal{H}_2$.

In the propostion below, we will show that the curves $\mathcal{H}_1^{(2)}$ and $\mathcal{H}_2^{(2)}$ can be defined over the field $\mathbb{F}_q$. This will be done by giving an explicit description of these curves over the field $\mathbb{F}_q$.

5.7 Proposition. *The following statements hold:*

i) *The curves $\mathcal{H}_0$, $\mathcal{H}_1^{(2)}$ and $\mathcal{H}_2^{(2)}$ can be defined over the field $\mathbb{F}_q$.*

ii) *There exists an element $g_{\mathcal{H}} \in GL(3, \mathbb{F}_{q^4})$ that defines an isomorphism $\mathcal{H}_0 \otimes \mathbb{F}_{q^4} \cong \mathcal{H}_1 \otimes \mathbb{F}_{q^4}$. In particular, the following statement holds:*

$$x \in \mathcal{H}_1 \otimes \mathbb{F}_{q^4} \text{ if and only if } g_{\mathcal{H}}(x) \in \mathcal{H}_0 \otimes \mathbb{F}_{q^4}.$$

iii) *The hermitian curves $\mathcal{H}_0$ and $\mathcal{H}_1$ are not isomorphic over $\mathbb{F}_{q^2}$.*

iv) *The hermitian curves $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic over $\mathbb{F}_{q^2}$.*

v) *$\mathcal{H}_1^{(2)} \otimes \mathbb{F}_{q^2} \cong \mathcal{H}_2^{(2)} \otimes \mathbb{F}_{q^2}$ and consists of two connected components isomorphic to $\mathcal{H}_1 \cong \mathcal{H}_2$.*

vi) *The curves $\mathcal{H}_1^{(2)}$ and $\mathcal{H}_2^{(2)}$ are isomorphic over $\mathbb{F}_q$.*

Proof. The curve $\mathcal{H}_0$ is already defined over $\mathbb{F}_q$. To prove statement (i) of the proposition, we only have to consider the curves $\mathcal{H}_1^{(2)}$ and $\mathcal{H}_2^{(2)}$. To prove it we consider the curve $X_{\omega^2} \otimes k$, $k \cong \mathbb{F}_q$, embedded in $\mathbb{P}_k^4$ and defined by the equations $y_0 = 0$, $y_1 y_2 - y_3 y_4 = 0$ and $y_1 y_3 - \omega^2 y_2 y_3 = 0$. Let us fix the notation and write $\widetilde{\omega}$ with $\widetilde{\omega}^q = -\widetilde{\omega}$ for the element $\omega \otimes k$ of $\mathbb{F}_{q^2}$. The variety $X_{\omega^2} \otimes k$ contains the points $y = (y_0, y_1, y_2, y_3, y_4) = (0, \epsilon\widetilde{\omega} y_2, y_2, \epsilon\widetilde{\omega} y_4, y_4)$ with $\epsilon = \pm 1$. The variety defined by the equation $y_0^{q+1} + y_2 y_3^q + y_2^q y_3 = 0$ with $(0, y_1, y_2, y_3, y_4) \in X_{\omega^2} \otimes k$ is a curve of type $\mathcal{H}_1^{(2)}$ and is defined over $k \cong \mathbb{F}_q$. Replacing y_3 by $y_4/\epsilon\widetilde{\omega}$, we obtain the equation $\epsilon\widetilde{\omega} y_0^{q+1} + (y_2^q y_4 - y_2 y_4^q) = 0$, where $\epsilon y_1 = \widetilde{\omega} y_2$ and $\epsilon y_3 = \widetilde{\omega} y_4$. This shows that the curve also equals a curve $\mathcal{H}_1^{(2)}$ defined over $k \cong \mathbb{F}_q$. This proves statement (i) and, moreover, shows that the curves $\mathcal{H}_1^{(2)}$ and $\mathcal{H}_2^{(2)}$ are isomorphic over $\mathbb{F}_q$. Therefore we have also proved statement (vi) of the lemma.

To prove the second statement of the proposition, we explicitly construct an element $g \in SL(3, \mathbb{F}_{q^4})$ that has the required properties.

Let $\widetilde{\beta}$ be an element in some extension of $\mathbb{F}_{q^2}$ such that $\widetilde{\beta}^{q+1} = \widetilde{\omega}$. Here $\widetilde{\omega} \in \mathbb{F}_{q^2}$ is such that $\widetilde{\omega}^{q-1} = -1$ holds. Then $\widetilde{\beta}^{q^2-1} = \widetilde{\omega}^{q-1} = -1$ and $\widetilde{\beta}^{q^4-1} = (-1)^{q^2+1} = 1$. Therefore we can choose $\widetilde{\beta} \in \mathbb{F}_{q^4}$. Let $g \in GL(3, \mathbb{F}_{q^4})$ be the diagonal element $g := diag(\widetilde{\beta}^{-1}, 1, \widetilde{\omega}^{-1})$. Let $x \in \mathcal{H}_1$ be the point (x_0, x_1, x_2). Then $g(x) = (\widetilde{\beta}^{-1} x_0, x_1, \widetilde{\omega}^{-1} x_2)$ satisfies the equation defining $\mathcal{H}_0$. The element $g_{\mathcal{H}} := g$ defines an isomorphism between the curves $\mathcal{H}_1 \otimes \mathbb{F}_{q^4}$ and $\mathcal{H}_0 \otimes \mathbb{F}_{q^4}$. This proves statement (ii) of the proposition.

To prove the third statement of the proposition, we consider the number of points on the hermitian curves $\mathcal{H}_0$ and $\mathcal{H}_1$ that are $\mathbb{F}_{q^2}$-valued. The curve $\mathcal{H}_1$ has q^3+1 points that are defined over $\mathbb{F}_{q^2}$. The only $\mathbb{F}_{q^2}$-valued points on the curve $\mathcal{H}_0$ are the $q+1$ points $y \in \mathcal{H}_0$ with $y_0 = 0$. These points are in fact defined over $\mathbb{F}_q$. It follows that there does not exist an element $g \in SL(3, \mathbb{F}_{q^2})$ such that $g(x) \in \mathcal{H}_0$ if and only if $x \in \mathcal{H}_1$. This proves statement (iii)

The isomorphism $\mathcal{H}_1 \cong \mathcal{H}_2$ over $\mathbb{F}_{q^2}$ can be defined by using the diagonal element $h = diag(1, \widetilde{\omega}, 1) \in GL(3, \mathbb{F}_{q^2})$, where $\widetilde{\omega}^{q-1} = -1$. Then $x \in \mathcal{H}_1$ if and only if $h(x) \in \mathcal{H}_2$. This proves statement (iv).

Statement (v) of the proposition follows directly from the definitions. Since we already have proved statement (vi), this completes the proof of the proposition.

□

5.8. Comparison with Teitelbaums description. In [D] Drinfel'd has defined a system of étale coverings of $\Omega_{1,K} := \mathbb{P}^1_K - \mathbb{P}^1(K)$. The first level $\Sigma_K^{(q-1)}$ of this system of coverings consists of $q-1$ connected components that are all isomorphic (See [Te]). The reduction of a connected component of this first level consists of hermitian curves. In fact, Teitelbaum derives over the field $\mathbb{C}_p$ the following equation for $\Sigma_K^{(q-1)}$ around a vertex in affine coordinates : $Y_0^{p^2-1} = p(z_0 - z_0^p)^{p-1}$ (See [Te] cor. 6). Here $p = q$, since only $K = \mathbb{Q}_p$ is considered in [Te].

The constant factor p matters for a definition over K° and is irrelevant for our construction. Taking a connected component Σ_K therefore gives $Y_0^{p+1} = z_0 - z_0^p$ in affine coordinates over $\mathbb{C}_p$. Below we will show that one can in fact construct a connected component Σ_K over K. Assuming this, the étale covering of $\Omega_{1,K}$ described in [Te] has a reduction that consists of hermitian curves $\mathcal{H}_0$ instead of curves $\mathcal{H}_1$. Therefore the étale covering differs from ours in the way it is defined over the field K.

Our methods can be used to construct the curve described by Teitelbaum over the field K. To define an affinoid $X_{\mathbf{e}_0}^{\Sigma_K}$ for the edge $\mathbf{e}_0 \in \mathbf{b}$, one considers $\widetilde{f}_{\mathbf{e}_0}(x) := \frac{x_1}{x_2} \cdot \frac{1-(\frac{x_1}{x_2})^{(q-1)}}{1-(-\frac{\pi \cdot x_2}{x_1})^{(q-1)}}$. Let $\widetilde{h}_{\mathbf{e}_0}(x)^{q+1} = -\widetilde{f}_{\mathbf{e}_0}(x)$. Here $\frac{x_1}{x_2} \in \Omega_{1,K}$, $1 \geq |\frac{x_1}{x_2}| \geq |\pi|$. Then the affinoids $X_{\mathbf{e}}^{\Sigma_K}$ for the edges $\mathbf{e} \in \mathbf{b}$ glue together into a rigid analytic variety Σ_K that is an étale covering of $\Omega_{1,K}$ of degree $q+1$. The methods used above to define the group action on Σ and describe the reduction of Σ are also applicable in this case.

One can construct the isomorphism $\Sigma_K \otimes L \cong \Sigma$ explicitly. Let $\varphi : \Omega_{1,K} \otimes L \to \Omega_1$ be the isomorphism given by $\varphi((x_1, x_2)) = (x_1, \omega x_2)$, with $\omega \in L^\circ$, $\omega^{q-1} = -1$. Then $\varphi_* f_{\mathbf{e}_0} = \omega \widetilde{f}_{\mathbf{e}_0}$. Therefore taking an element $\beta \in L^\circ$ such that $\beta^{q+1} = \omega$, we can take $\varphi_* h_{\mathbf{e}_0} = \beta \widetilde{h}_{\mathbf{e}_0}$ and obtain an isomorphism between the affinoids $X_{\mathbf{e}_0}^{\Sigma}$ and $X_{\mathbf{e}_0}^{\Sigma_K} \otimes L$ that is defined over the unramified extension $L' \supset K$ of degree four. This isomorphism corresponds to the isomorphism between the hermitian curves $\mathcal{H}_0 \otimes \mathbb{F}_{q^4}$ and $\mathcal{H}_1 \otimes \mathbb{F}_{q^4}$ given above.

One can also define a variety $\Omega_{1,K}^{(q-1)}$ that consists of $q-1$ connected components all isomorphic to $\Omega_{1,K}$. The equations $(y_1 y_3)^{\frac{q-1}{2}} = (y_2 y_4)^{\frac{q-1}{2}}$ and $y_2 y_3 - y_1 y_4 = 0$ in $\mathbb{P}_K^3$ define a variety that consists of $q-1$ connected components that are all isomorphic to $\mathbb{P}_K^1$. Removing the K-rational points from this variety gives the variety $\Omega_{1,K}^{(q-1)}$. Using the variety $\Omega_{1,K}^{(q-1)}$, the construction of the variety $\Sigma_K^{(q-1)}$ is similar to that of $\Sigma^{(q-1)}$.

As a consequence there exist two distinct $SL(2, K)$-equivariant finite étale coverings $\Sigma^{(q-1)}$ and $\Sigma_K^{(q-1)}$ of $\Omega_{1,K} = \mathbb{P}_K^1 - \mathbb{P}^1(K)$ that are defined over the field K. These spaces are isomorphic over some finite unramified extension of the field K. The following statements hold:

i) $\Sigma^{(q-1)} \otimes L' \cong \Sigma_K^{(q-1)} \otimes L'$. Here $L' \supset K$ is the unramified extension of degree four.

ii) a) The connected components of $\Sigma^{(q-1)}$ are defined over L and not over K.

  b) The connected components of $\Sigma_K^{(q-1)}$ are defined over K.

iii) a) The Galois group $Gal(L/K)$ fixes no connected component of $\Sigma^{(q-1)}$.

  b) The Galois group $Gal(L/K)$ fixes each connected component of $\Sigma_K^{(q-1)}$.

iv) a) The reduction of $\Sigma^{(2)}$ consists of curves $\mathcal{H}_1^{(2)}$.

  b) The reduction of Σ_K consists of curves $\mathcal{H}_0$.

Consequently, the étale maps $\Sigma^{(q-1)} \to \Omega_{1,K}$ and $\Sigma_K^{(q-1)} \to \Omega_{1,K}$ are different over the field K.

5.9. Frobenius action. Let $K^{nr} \supset K$ denote the maximal unramified extension of K and let $F: K^{nr} \to K^{nr}$ denote the Frobenius map. For any (analytic) variety X that is defined over K, we have a well-defined action of the Frobenius map on the space $X \otimes K^{nr} := X \otimes_K K^{nr}$.

We will abuse the notation a little bit in the sense that on $X_K \otimes K^{nr}$ the Frobenius map will be the one that corresponds to the definition of X_K over K. Therefore even though $X_K \otimes K^{nr} \cong X'_K \otimes K^{nr}$ may hold, the Frobenius maps we use on these spaces are different if $X_K \not\cong X'_K$ over K. For instance on ${\Omega_1}^{(2)} \otimes K^{nr}$ we use the Frobenius map that permutes the two connected components of ${\Omega_1}^{(2)} \otimes K^{nr}$, whereas on the isomorphic space $\Omega_{1,K}^{(2)} \otimes K^{nr}$ we use the Frobenius map that preserves both connected components.

5.10. Group action combined with Frobenius action. Let X be an (analytical) variety defined over K. Let $Aut_K(X)$ be the the group of automorphisms of X, whose action is defined over K. We assume that $Aut_K(X) \subset GL(n, K)$. Therefore we have a well-defined determinant function on $Aut_K(X)$.

We define an action of $Aut_K(X)$ on $X \otimes K^{nr}$ by taking $g(x \otimes y) = g(x) \otimes F^{-v(det(g))}(y)$ for $g \in Aut_K(X)$, $x \in X$ and $y \in K^{nr}$. By this we mean that the action of the element $g \in Aut_K(X)$ on a point $z \in X(K^{nr})$ is given by: $g(F^{-v(det(g))}(z)) = F^{-v(det(g))}(g(z))$, where $g(z)$ denotes the usual action of the element $g \in Aut_K(X)$.

5.11. Example. We continue with the example above to indicate how the action of a discrete group Δ differs on the analytic spaces $\Sigma_K^{(q-1)}$ and $\Sigma^{(q-1)}$. We let Δ act on the analytic varieties $\Sigma_K^{(q-1)} \otimes K^{nr}$ and $\Sigma^{(q-1)} \otimes K^{nr}$ by $\gamma(x \otimes y) = \gamma(x) \otimes F^{-v(det(\gamma))}(y)$, where $x \in \Sigma_K^{(q-1)}, \Sigma^{(q-1)}$ and $y \in K^{nr}$.

The quotients by the group Δ_0 are defined over K^{nr}. One has $(\Sigma_K^{(q-1)} \otimes K^{nr})/\Delta_0 \cong (\Sigma_K^{(q-1)}/\Delta_0) \otimes K^{nr} \cong (\Sigma^{(q-1)} \otimes K^{nr})/\Delta_0 \cong (\Sigma^{(q-1)}/\Delta_0) \otimes K^{nr}$. Here $\Sigma_K^{(q-1)}/\Delta_0$ and $\Sigma^{(q-1)}/\Delta_0$ denote the quotients using the usual linear action of the group Δ_0 on $\Sigma_K^{(q-1)}$ and $\Sigma^{(q-1)}$, respectively. The isomorphic quotients above consist of $q-1$ connected components isomorphic to $(\Sigma_K \otimes K^{nr})/\Delta_0$. Each connected component has a reduction that consists of $m_\Delta = \sharp\{\mathbf{v} \mid \mathbf{v} \in \mathbf{b}/\Delta_0\}$ hermitian curves (defined over the separable algebraic closure of ℓ).

Since the element $\gamma_2^{-1} \in \Delta$ acts as F^2 on the spaces, the quotients by the group Δ_2 are defined over the field L. One has $(\Sigma_K^{(q-1)} \otimes K^{nr})/\Delta_2 \cong (\Sigma_K^{(q-1)}/\Delta_0) \otimes L$ and $(\Sigma^{(q-1)} \otimes K^{nr})/\Delta_2 \cong (\Sigma^{(q-1)}/\Delta_0) \otimes L$. These quotients consist of $q-1$ connected components that are defined over L. Each connected component has a reduction that consists of m_Δ hermitian curves of type $\mathcal{H}_0 \otimes \mathbb{F}_{q^2}$ and of type $\mathcal{H}_1$, respectively. Therefore these quotients are not isomorphic over L.

The quotients $(\Sigma_K^{(q-1)} \otimes K^{nr})/\Delta$ and $(\Sigma^{(q-1)} \otimes K^{nr})/\Delta$ are both defined over the field K. The quotient $(\Sigma_K^{(q-1)} \otimes K^{nr})/\Delta$ consists of $q-1$ connected components $(\Sigma_K \otimes K^{nr})/\Delta$ that are all defined over K. Each connected component has a reduction consisting of $m_\Delta/2$ hermitian curves $\mathcal{H}_0$.

The quotient $(\Sigma^{(q-1)} \otimes K^{nr})/\Delta$ consists of $(q-1)/2$ connected components $(\Sigma^{(2)} \otimes K^{nr})/\Delta$ that are defined over K. Each connected component has a reduction consisting of $m_\Delta/2$ curves $\mathcal{H}_1^{(2)}$. The two connected components of a curve $\mathcal{H}_1^{(2)}$ belong to two different vertices $\mathbf{v} \in \mathbf{b}/\Delta_0$. These vertices form an orbit for the action of the group Δ/Δ_2 on $\mathbf{b}/\Delta_0$.

Similar properties hold for the action of Δ on ${\Omega_1}^{(q-1)} \otimes K^{nr}$ and $\Omega_{1,K}^{(q-1)} \otimes K^{nr}$. Then the following holds:

i) The quotient $({\Omega_1}^{(q-1)} \otimes K^{nr})/\Delta$ consists of $(q-1)/2$ connected components $({\Omega_1}^{(2)} \otimes K^{nr})/\Delta$ that are defined over K. The reduction of a connected component consists of $m_\Delta/2$ components isomorphic to curves $X_{\omega^2} \otimes k$. The connected components $\mathbb{P}^1_\ell$ of such a curve $X_{\omega^2} \otimes \ell$ belong to two vertices of $\mathbf{b}/\Delta_0$ that form a single orbit of the group Δ/Δ_2.

ii) The quotient $(\Omega_{1,K}^{(q-1)} \otimes K^{nr})/\Delta$ consists of $q-1$ connected components $(\Omega_{1,K} \otimes K^{nr})/\Delta$ that are defined over K. The reduction of a connected component consists of $m_\Delta/2$ components isomorphic to curves $\mathbb{P}^1_k$.

We have a map $(\Omega_{1,K}^{(q-1)} \otimes K^{nr})/\Delta \to (\Omega_{1,K} \otimes K^{nr})/\Delta$ and a map $({\Omega_1}^{(q-1)} \otimes K^{nr})/\Delta \to (\Omega_{1,K} \otimes K^{nr})/\Delta$. The first map is just a projection to a connected component. The second map also applies an automorphism of order two to each of the connected components of $({\Omega_1}^{(q-1)} \otimes K^{nr})/\Delta$. This automorphism identifies the connected components of the curves $X_{\omega^2} \otimes k$ in the reduction. It therefore acts as Δ/Δ_2 on $\mathbf{b}/\Delta_2 = \mathbf{b}/\Delta_0$.

5.12. Drinfel'ds étale system. Let $D \supset K$ be a central division algebra with invariant $1/d$ and let $\mathcal{O}_D \subset D$ be the maximal order. Let $L_d \supset K$ be the unramified extension of degree d. Then $D = L_d[\Pi]/(\Pi^d - \pi)$ and $\mathcal{O}_D = L_d^\circ[\Pi]/(\Pi^d - \pi)$. The element $\Pi \in D$ acts on elements $a \in L_d$ as $\Pi a = \sigma(a)\Pi$, where σ is a generator of $Gal(L_d/K)$.

Let $\Omega_{d-1,K}$ be the analytic variety $\Omega_{d-1,K} := \mathbb{P}_K^{d-1} - \{K-\text{rational hyperplanes}\}$. Drinfel'd has constructed a tower of étale coverings of $\Omega_{d-1,K} \otimes K^{nr}$ using moduli. The Galois group of this sytem of étale coverings is the group $\mathcal{O}_D^*$. The n-th level of this system is an étale covering of $\Omega_{d-1,K} \otimes K^{nr}$ with Galois group $(\mathcal{O}_D/\Pi^n\mathcal{O}_D)^*$.

Drinfel'd defines an action of the Frobenius map $F : K^{nr} \to K^{nr}$ on the sytem of étale coverings. This constitutes a descent datum and implies a definition of the spaces over the field K. He defines an action of the group $GL(d,K)$ on both the sytem of étale coverings and on $\Omega_{d-1,K} \otimes K^{nr}$. The action of $GL(d,K)$ and of the Frobenius map F are combined using the determinant function. On $\Omega_{d-1,K} \otimes K^{nr}$ the action of an element $g \in GL(d,K)$ is given by letting an element g act through $g \otimes F^{-v(det(g))}$. The analytical spaces in the étale system are defined over K^{nr} but through the action of the group $GL(d,K)$ combined with the Frobenius map one does have an implicit definition of these spaces over the field K.

Drinfel'd only considers the levels md, $m \in \mathbb{Z}$. These correspond to the Galois groups $(\mathcal{O}_D/\pi^m\mathcal{O}_D)^* = (\mathcal{O}_D/\Pi^{md}\mathcal{O}_D)^*$. Since the covering $\Sigma^{(q-1)} \otimes K^{nr} \cong \Sigma_K^{(q-1)} \otimes K^{nr}$ correspond to the level with Galois group $(\mathcal{O}_D/\Pi\mathcal{O}_D)^* = (L_d^\circ/\pi L_d^\circ)^*$ with $d = 2$, they are not officially part of Drinfel'ds étale system.

5.13. Comparison with Drinfel'ds étale system. Taking the higher étale coverings of $\Omega_{1,K}$ and assuming for convenience that they can all be defined over K above Σ_K, then these higher étale coverings can also be defined over K in two different ways. In particular, the inverse limit Σ^∞ of the system of étale coverings can be defined over K in two different ways. This gives two different $SL(2,K)$-equivariant maps $\Sigma^\infty \to \Omega_{1,K}$ that are defined over K.

The proofs of Drinfel'd are based on moduli and purely functorial. Both the group action and the Frobenius map on $\Sigma^\infty \otimes K^{nr}$ are defined using the interpretation of the space as a moduli space. I do not know which of the possible definitions of Σ^∞ over K corresponds to his description of Σ^∞ via moduli. However, in theorem 8 of [Te] it is shown that the quotient of the first level of the étale system by a discrete torsion free co-compact subgroup $\Gamma \subset GL(2,K)$ consists of $q-1$ connected components. Therefore the Frobenius map F used is such that the implied definition over K corresponds to $\Sigma_K^{(q-1)}$. In particular, each connected component is defined over the field K itself.

Our construction generalises to higher dimensions. Let $V_d \cong L_d^d$ be a vector space defined over the field L_d. We take a L_d-basis $e_0, \ldots, e_{d-1}$ of

V_d, such that a generator σ of $Gal(L_d/K)$ acts by $\sigma(e_i) = e_{i+1}$, where the subscripts are read modulo d. Let the group $SL(d, K)$ act linear on the vector space V_d such that some maximal K-split torus $T \subset SL(d, K)$ acts diagonally on V_d with respect to the basis e_i, $i = 0, \ldots, d-1$. Let $\mathcal{L}_d \subset \mathbb{P}(V_d) \cong \mathbb{P}^{d-1}_{L_d}$ be the set of points that correspond to the orbit $SL(d, K) \cdot e_0$. Let $\mathcal{H}_d \subset \mathbb{P}^{d-1}_{L_d}$ be the set consisting of the hyperplanes in $\mathbb{P}^{d-1}_{L_d}$ that are spanned by $d-1$ points of $\mathcal{L}_d$.

Let $\Omega_{d-1} \subset \mathbb{P}^{d-1}_{L_d}$ be a rigid analytical variety $\Omega_{d-1} := \mathbb{P}^{d-1}_{L_d} - \mathcal{H}_d$. The group $SL(d, K)$ acts on Ω_{d-1} and $\Omega_{d-1} \cong \Omega_{d-1,K} \otimes L_d$. Let $\Omega^{(d)}_{d-1}$ be a variety defined over the field K, such that $\Omega^{(d)}_{d-1} \otimes L_d$ consists of d connected components isomorphic to Ω_{d-1} and such that the Galois group $Gal(L_d/K)$ permutes the d connected components of $\Omega^{(d)}_{d-1} \otimes L_d$. We have an $SL(d, K)$-equivariant étale map $\Omega^{(d)}_{d-1} \to \Omega_{d-1,K}$ that is defined over K.

The first level of Drinfel'ds étale system has Galois group $(\mathcal{O}_D/\Pi\mathcal{O}_D)^* = (L^\circ_d/\pi L^\circ_d)^* \cong \mu_{q^d-1}$. Here μ_{q^d-1} is the group consisting of the q^d-1-th roots of unity in L°. The elements $\zeta \in \mu_{q^d-1}$ with norm $N_{L_d/K}(\zeta) = 1$ are precisely the elements of the Galois group that preserve the connected components of the étale covering. Therefore the covering consists of $q-1$ connected components.

Presumably these connected components can be defined over K and form an étale covering of $\Omega_{d-1,K}$. Let us denote such a connected component by $\Sigma_{d-1,K}$ and the étale covering by $\Sigma^{(q-1)}_{d-1,K}$. One can define an étale covering Σ_{d-1} of Ω_{d-1} over the field L_d, that is analogous to the covering $\Sigma_{d-1,K} \to \Omega_{d-1,K}$. Then $\Sigma_{d-1,K} \otimes L'_d \cong \Sigma_{d-1} \otimes L'_d$ for some finite unramified extension $L'_d \supset L_d$. The covering $\Sigma_{d-1} \to \Omega_{d-1}$ can be used to define a covering $\Sigma^{(d)}_{d-1}$ of $\Omega^{(d)}_{d-1}$ that is defined over K. Here $\Sigma^{(d)}_{d-1}$ is such that $\Sigma^{(d)}_{d-1} \otimes L_d$ consists of d isomorphic connected components. If $d \mid q-1$, then $(q-1)/d$ copies of $\Sigma^{(d)}_{d-1}$ can be used to form a different étale covering $\Sigma^{(q-1)}_{d-1}$ of $\Omega_{d-1,K}$. Therefore $d \mid q-1$ must hold. Drinfel'd has no restrictions on the order q of the residue field of K. This again shows that his action of Frobenius implies a definition of the étale system over the field K such that each connected component of the étale system is defined over the field K itself. Therefore the first level is $\Sigma^{(q-1)}_{d-1,K}$.

Let us now compare the Frobenius maps on $\Sigma^{(q-1)}_{d-1} \otimes K^{nr}$ and $\Sigma^{(q-1)}_{d-1,K} \otimes K^{nr}$. Our Frobenius acts on the maximal K-split torus T that acts diagonally w.r.t. the basis e_i, $i = 0, \ldots, d$ of V_d as an element w of order d in the Weyl group of $SL(d, K)$. This element w is a Coxeter element of the Weyl group. While Drinfel'd uses the standard Frobenius F, our construction involves the Frobenius wF, i.e. the standard Frobenius twisted by a Coxeter element of the Weyl group of $SL(d, K)$.

6 An equivariant embedding into the projective plane

Let $\Gamma \subset SU(2, L)$ be a discrete co-compact subgroup. Let $\mathbb{P}$ and $\mathbb{P}^\vee$ be two projective planes $\mathbb{P}^2_L$ on which the group $SU(2, L)$ act linearly such that it fixes a single point. The action of $SU(2, L)$ on $\mathbb{P}$ and $\mathbb{P}^\vee$ differs by conjugation with the non-trivial element of the Galois group $Gal(L/K)$.

We embed the open admissable subspaces $\Sigma_{\mathbf{v}} \subset \Sigma$, $\mathbf{v} \in \mathbf{b}$ with $\tau(\mathbf{v}) = 0$ into $\mathbb{P}$ and the subspaces $\Sigma_{\mathbf{v}}$ with $\tau(\mathbf{v}) = 1$ into $\mathbb{P}^\vee$. These embeddings are defined using infinite Γ-invariant sums that converge on Ω_1. The embeddings therefore explicitly use the fact that Σ is a covering of Ω_1. Moreover, the embeddings are Γ-equivariant. The spaces $\Sigma_{\mathbf{v}}$, $\mathbf{v} \in \mathbf{b}$ are glued along the spaces $\Sigma_{\mathbf{e}}$ for edges $\mathbf{e} \in \mathbf{b}$ to obtain the analytic space Σ.

6.1. Infinite sums on Ω_1. Let $\Gamma \subset SU(2, L)$ be a discrete co-compact subgroup. We define the Γ-invariant infinite sums converging on Ω_1 that can be used to obtain a Γ-equivariant embedding of $\bigcup_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=0} \Sigma_{\mathbf{v}}$ ($\bigcup_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=1} \Sigma_{\mathbf{v}}$) into $\mathbb{P}^2_L$.

Let $\mathbf{v}_0 \in \mathbf{b}$ be the vertex corresponding to the equivalence class $[M_{\mathbf{v}_0}]$, $M_{\mathbf{v}_0} =< e_0, e_1, e_2 >$ and let $\mathbf{v}_1 \in \mathbf{b}$ be the vertex corresponding to the equivalence class $[M_{\mathbf{v}_1}]$, $M_{\mathbf{v}_1} =< e_0, e_1, \pi^{-1}e_2 >$. We associate to the vertices $\mathbf{v}_0, \mathbf{v}_1 \in \mathbf{b}$ polynomials $A_{\mathbf{v}_0}(x)$ and $A_{\mathbf{v}_1}(z)$ of degree $q+1$. For points $x, z \in \Omega_1$ one defines $A_{\mathbf{v}_0}(x) := x_1x_2^q + x_2x_1^q$ and $A_{\mathbf{v}_1}(z) := z_1(\pi z_2)^q + (\pi z_2)z_1^q$. The action of the group $SU(2, L)$ on the coordinates x_i and z_i, $i = 1, 2$, differs by a conjugation with the non-trivial element of the Galois group $Gal(L/K)$.

We associate to each vertex $\mathbf{v} \in \mathbf{b}$ a homogeneous polynomial $a_{\mathbf{v},\mathbf{b}}(x)$ or $a_{\mathbf{v},\mathbf{b}}(z)$ of degree $q + 1$ such that the following four conditions hold:

i) If $\tau(\mathbf{v}) = 0$, then $a_{\mathbf{v},\mathbf{b}}(x) \equiv g^*A_{\mathbf{v}_0}(x) \mod \pi$ for all $g \in SU(2, L)$ such that $g(\mathbf{v}_0) = \mathbf{v}$.

ii) If $\tau(\mathbf{v}) = 1$, then $a_{\mathbf{v},\mathbf{b}}(z) \equiv g^* A_{\mathbf{v}_1}(z) \mod \pi$ for all $g \in SU(2, L)$ such that $g(\mathbf{v}_1) = \mathbf{v}$.

iii) If $\tau(\mathbf{v}) = 0$, $a_{\gamma(\mathbf{v}),\mathbf{b}}(x) = \gamma^* a_{\mathbf{v},\mathbf{b}}(x)$ for all $\gamma \in \Gamma$.

iv) If $\tau(\mathbf{v}) = 1$, $a_{\gamma(\mathbf{v}),\mathbf{b}}(z) = \gamma^* a_{\mathbf{v},\mathbf{b}}(z)$ for all $\gamma \in \Gamma$.

Let $F_{\mathbf{b}}(x) := (\sum_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=0} a_{\mathbf{v},\mathbf{b}}(x)^{-1})^{-1}$ and let $F_{\mathbf{b}}^{\vee}(z) := (\sum_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=1} a_{\mathbf{v},\mathbf{b}}(z)^{-1})^{-1}$. Below we show that these sums are well-defined for points $x, z \in \Omega_1$.

Let $A \subset \mathbf{b}$ be an apartment and let x_1, x_2 be the associated coordinates and let $x_1 x_2$ be the torus invariant belonging to the apartment A. To the apartment $g(A) \subset \mathbf{b}$, $g \in SU(2, L)$ belongs the invariant $g^* x_1 g^* x_2$. Since the characteristic of the residue field of K is greater than two, there exists a Γ-invariant complete transversal system of apartments $\mathcal{T}_{\mathbf{b}}(A)$. So we can define the sum $Z_{\mathbf{b}}(x) := (\sum_{gA\in\mathcal{T}_{\mathbf{b}}(A)} (g^* x_1 g^* x_2)^{-1})^{-1}$.

6.2 Lemma. *The following statements hold:*

i) Let $x \in \Omega_1$ be a point and let $A \subset \mathbf{b}$ be an apartment containing $\psi(x)$. Then $v(\frac{g^* x_1 g^* x_2}{x_1 x_2}(x)) = -d_{\mathbf{b}}(gA, \psi(x))$.

ii) The sum $\sum_{gA\in\mathcal{T}_{\mathbf{b}}(A)} (g^* x_1 g^* x_2)^{-1}$ converges on Ω_1.

iii) The sum $\sum_{gA\in\mathcal{T}_{\mathbf{b}}(A)} (g^* x_1 g^* x_2)^{-1}$ has only zeroes at points $x \in \Omega_1$ such that $\psi(x)$ is a vertex $\mathbf{v} \in \mathbf{b}$.

Proof. Let us prove statement (i) of the lemma. If $\psi(x) \in gA$, then $d_{\mathbf{b}}(gA, \psi(x)) = 0$ and we may assume that $g \in P_{\psi(x)}$. Then $|\frac{g^* x_i}{x_i}| = 1$ for $i = 1, 2$ and statement (i) of the lemma holds.

So let us now consider the case where $\psi(x) \notin gA$. Let $\mathbf{v} \in gA$ be the vertex closest to $\psi(x)$. Let $\mathbf{e} \in A \subset \mathbf{b}$ be the edge containing $\psi(x)$. Without loss of generality, we may assume that $\mathbf{e}$ is our standard edge. Since $\forall (h \in P_{\mathbf{e}})\ |\frac{h^* x_i}{x_i}| = 1$, $i = 1, 2$, the absolute value of the coordinates x_1, x_2 does not depend on the choice of the apartment $A \ni \psi(x)$. Therefore we may assume that the vertex $\mathbf{v}$ is contained in A.

Let us first consider the case where the vertex $\mathbf{v}$ is of type $\tau(\mathbf{v}) = 0$. Let $[M_{\mathbf{v}}]$ be a the equivalence class of L°-modules corresponding to the vertex $\mathbf{v}$. We may assume that $M_{\mathbf{v}} =< e_1, e_2 >$ and after replacing the the coordinates x_i by suitable translates by an element of the maximal split torus belonging to the apartment A, we may assume that the coordinates x_i, $i = 1, 2$, belong to the basis e_i of $\mathbb{P}^1_L$.

We normalise the coordinates of the point x such that $|x_i| \leq 1$ and one of the coordinates has absolute value equal to 1. Using this normalisation the equality $v(x_1x_2) = d_{\mathbf{b}}(\mathbf{v}, \psi(x))$ holds.

Since the apartment $g(A)$ is such that the vertex $\mathbf{v} \in g(A)$ is closest to the point $\psi(x) \in A$, $|g^*x_i| = 1$ holds for $i = 1, 2$. Therefore $-v(\frac{g^*x_1g^*x_2}{x_1x_2}) = v(x_1x_2) = d_{\mathbf{b}}(\mathbf{v}, \psi(x))$ for elements $g \in P_{\mathbf{v}}$ such that $g(A) \bigcap A = \{\mathbf{v}\}$.

This proves statement (i) of the lemma if the vertex $\mathbf{v}$ is of type $\tau(\mathbf{v}) = 0$. The proof in case the vertex $\mathbf{v}$ is of type $\tau(\mathbf{v}) = 1$ is similar.

By statement (i) of the lemma $|(g^*x_1g^*x_2)^{-1}| \to 0$ if $d_{\mathbf{b}}(gA, \psi(x)) \to \infty$. Statement (ii) of the lemma follows from this and the fact that only finitely many apartments $gA \in \mathcal{T}_{\mathbf{b}}(A)$ have a distance $d_{\mathbf{b}}(gA, \psi(x)) < R$ for any finite value of $R \in \mathbb{R}$.

A zero of the sum $\sum_{gA \in \mathcal{T}_{\mathbf{b}}(A)} (g^*x_1g^*x_2)^{-1}$ can only occur, if more than one of the polynomials $g^*x_1g^*x_2$ obtains the minimal absolute value of such polynomials on x. Hence more than a single apartment $gA \in \mathcal{T}_{\mathbf{b}}(A)$ should be of minimum distance to the point $\psi(x)$. This can only occur if $\psi(x)$ is a vertex $\mathbf{v} \in \mathbf{b}$. This proves statement (iii) of the lemma. □

6.3 Lemma. *Let $x \in X_{\mathbf{e}} \subset \Omega_1$ be a point and let $\mathbf{v} \in \mathbf{b}$ be a vertex. Let $A \subset \mathbf{b}$ be an apartment containing $\psi(x)$ and let x_1, x_2 be the associated coordinates. Then:*

i) If $\tau(\mathbf{v}) = 0$ and $d_{\mathbf{b}}(\mathbf{v}, \psi(x)) \leq 1$, then $v(\frac{a_{\mathbf{v},\mathbf{b}}(x)}{(x_1x_2)^{(q+1)/2}}) = -\frac{q-1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi(x))$.

ii) If $\tau(\mathbf{v}) = 0$ and $d_{\mathbf{b}}(\mathbf{v}, \psi(x)) > 1$, then $-\frac{q-1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi(x)) \geq v(\frac{a_{\mathbf{v},\mathbf{b}}(x)}{(x_1x_2)^{(q+1)/2}}) \geq -\frac{q+1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi(x)) + 1$.

iii) If $\tau(\mathbf{v}) = 1$ and $d_{\mathbf{b}}(\mathbf{v}, \psi^\vee(z)) \leq 1$, then $v(\frac{a_{\mathbf{v},\mathbf{b}}(z)}{(\pi z_1z_2)^{(q+1)/2}}) = -\frac{q-1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi^\vee(z))$.

iv) If $\tau(\mathbf{v}) = 1$ and $d_{\mathbf{b}}(\mathbf{v}, \psi^\vee(z)) > 1$, then $-\frac{q-1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi^\vee(z)) \geq v(\frac{a_{\mathbf{v},\mathbf{b}}(z)}{(\pi z_1z_2)^{(q+1)/2}}) \geq -\frac{q+1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi^\vee(z)) + 1$.

Proof. Let $\mathbf{e} \in A \subset \mathbf{b}$ be the edge containing $\psi(x)$. Without loss of generality, we may assume that $\mathbf{e}$ is our standard edge. Since $\forall (g \in P_{\mathbf{e}})\ |\frac{g^*x_i}{x_i}| = 1$, $i = 1, 2$, the absolute value of the coordinates x_1, x_2 does not depend on the choice of the apartment $A \ni \psi(x)$. Therefore we may assume that the vertex $\mathbf{v}$ is contained in A.

Let us first consider the case where the vertex $\mathbf{v}$ is of type $\tau(\mathbf{v}) = 0$. Since the characteristic of the residue field ℓ of L does not equal 2, the homogeneous polynomial $a_{\mathbf{v},\mathbf{b}}(x)$ of degree $q+1$ is such that $a_{\mathbf{v},\mathbf{b}}(x) \equiv (g_0^* x_1 g_0^* x_2) \cdot (g_1^* x_1 g_1^* x_2) \cdots (g_{(q-1)/2}^* x_1 g_{(q-1)/2}^* x_2) \mod \pi$ for suitably choosen elements $g_i \in P_{\mathbf{v}},\ i = 0, \ldots, (q-1)/2$. One can choose the elements g_i such that $g_i(A) \bigcap A = \{\mathbf{v}\}$ for $i = 1, \ldots, (q-1)/2$ and $g_0(A) \bigcap A$ contains the two edges $\mathbf{e} \ni \mathbf{v}$ that are contained in A.

By the lemma above $-v(\frac{g^* x_1 g^* x_2}{x_1 x_2}) = d_{\mathbf{b}}(\mathbf{v}, \psi(x))$ for $g \in P_{\mathbf{v}}$ such that $g(A) \bigcap A = \{\mathbf{v}\}$. If $\psi(x) \in g_0(A)$, then $-v(\frac{g_0^* x_1 g_0^* x_2}{x_1 x_2}) = 0$. Therefore $v(\frac{a_{\mathbf{v},\mathbf{b}}(x)}{(x_1 x_2)^{(q+1)/2}}) = -\frac{q-1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi(x))$, if $\psi(x) \in g_0(A)$. In particular, this holds if $d_{\mathbf{b}}(\mathbf{v}, \psi(x)) \leq 1$. This proves statement (i) of the lemma.

Statement (ii) of the lemma follows from the fact that $0 \leq d_{\mathbf{b}}(g_0 A, \psi(x)) \leq d_{\mathbf{b}}(\mathbf{v}, \psi(x)) - 1$, since the apartment $g_0 A$ contains the two edges $\mathbf{e} \ni \mathbf{v}$ that are contained in $A \ni \psi(x)$.

The proof of the statements (iii) and (iv) for the case $\tau(\mathbf{v}) = 1$ is similar. □

6.4 Lemma. *The following statements hold:*

i) The sums $\sum_{\mathbf{v} \in \mathbf{b}, \tau(\mathbf{v})=0} a_{\mathbf{v},\mathbf{b}}(x)^{-1}$ *and* $\sum_{\mathbf{v} \in \mathbf{b}, \tau(\mathbf{v})=1} a_{\mathbf{v},\mathbf{b}}(z)^{-1}$ *converge on* Ω_1.

ii) The sum $\sum_{\mathbf{v} \in \mathbf{b}, \tau(\mathbf{v})=0} a_{\mathbf{v},\mathbf{b}}(x)^{-1}$ *has only zeroes at points* $x \in \Omega_1$ *such that* $\psi(x)$ *is a vertex* $\mathbf{v}$ *of type* $\tau(\mathbf{v}) = 1$.

iii) The sum $\sum_{\mathbf{v} \in \mathbf{b}, \tau(\mathbf{v})=1} a_{\mathbf{v},\mathbf{b}}(z)^{-1}$ *has only zeroes at points* $z \in \Omega_1$ *such that* $\psi^{\vee}(z)$ *is a vertex* $\mathbf{v}$ *of type* $\tau(\mathbf{v}) = 0$.

Proof. From the previous lemma it follows that $a_{\mathbf{v},\mathbf{b}}(x)^{-1} \longrightarrow 0$, if $d_{\mathbf{b}}(\mathbf{v}, \psi(x)) \longrightarrow \infty$. Since there are only finitely many vertices $\mathbf{v} \in \mathbf{b}$, such that $d_{\mathbf{b}}(\mathbf{v}, \psi(x)) \leq R$ for any finite $R \in \mathbb{R}$, it follows that the first sum converges on all of Ω_1. The argument for the second sum is similar. This proves statement (i) of the lemma.

A zero of the sums can only occur, if more than one polynomial $a_{\mathbf{v},\mathbf{b}}(x)$ obtains the minimal absolute value of such polynomials on x. Hence more than one vertex $\mathbf{v}$ should obtain the minimum distance to the point $\psi(x) \in \mathbf{b}$. This can only occur if $\psi(x)$ is a vertex such that the type $\tau(\mathbf{v})$ is distinct from the type of the vertices used to define the sums. This proves the second statement of the lemma.

The third statement of the lemma is proved analogously. This concludes the proof of the lemma. □

6.5. The spaces $Y^s_{\mathbf{b}}$ and $Y^{s\vee}_{\mathbf{b}}$. Let us take a projective space $\mathbb{P}^5_L$ with homogeneous coordinates $(x_0, x_1, x_2, z_0, z_1, z_2)$ with a quadratic form $x_0z_0 + x_1z_2 + x_2z_1$ on it. In $\mathbb{P}^5_L$ one has two projective planes $\mathbb{P}^2_L$, one defined by $z_0 = z_1 = z_2 = 0$ and one defined by $x_0 = x_1 = x_2 = 0$. Then (x_0, x_1, x_2) and (z_0, z_1, z_2) are the homogeneous coordinates on these two distinct projective planes $\mathbb{P}^2_L$. Let us denote these projective planes by $\mathbb{P}$ and $\mathbb{P}^\vee$,respectively.

Let the unitary group $SU(2, L)$ act linearly on $\mathbb{P}^5_L$ preserving the quadratic form, the subspaces $\mathbb{P}$ and $\mathbb{P}^\vee$ and the points x_0 and z_0.

On $\mathbb{P}$ and $\mathbb{P}^\vee$ the group $SU(2, L)$ acts linearly fixing the point x_0 and z_0, respectively. The group $SU(2, L)$ acts linearly on the projective line $\mathbb{P}^1_L$ given by $x_0 = 0$ and $z_0 = 0$ in $\mathbb{P}$ and $\mathbb{P}^\vee$, respectively. The action of $SU(2, L)$ on the projective planes $\mathbb{P}$ and $\mathbb{P}^\vee$ differs by a conjugation with the non-trivial element of the Galois group $Gal(L/K)$.

Let us define hermitian forms h and $h^\vee$ preserved by $SU(2, L)$ on $\mathbb{P}$ and $\mathbb{P}^\vee$, respectively. Let $h(x, y) = x_1\overline{y}_2 + x_2\overline{y}_1 + x_0\overline{y}_0$ for $x, y \in \mathbb{P}$ and let $h^\vee(z, u) = z_1\overline{u}_2 + z_2\overline{u}_1 + z_0\overline{u}_0$ for $z, u \in \mathbb{P}^\vee$, where the points y and u are L-valued.

Let $Y^s_{\mathbf{b}} \subset \mathbb{P}$ denote the space $Y^s_{\mathbf{b}} := \{x \in \mathbb{P} \mid (x_1, x_2) \in \Omega_1\}$ and let $Y^{s\vee}_{\mathbf{b}} := \{z \in \mathbb{P}^\vee \mid (z_1, z_2) \in \Omega_1\}$. The spaces $Y^s_{\mathbf{b}}$ and $Y^{s\vee}_{\mathbf{b}}$ can be described using the hermitan forms h and $h^\vee$, respectively. Indeed, $Y^s_{\mathbf{b}} = \{x \in \mathbb{P} \cong \mathbb{P}^2_L \mid \forall (y = (0, y_1, y_2) \in \mathbb{P}^2(L) \text{ such that } h(y, y) = 0)\ h(x, y) \neq 0\}$. A similar description holds for $Y^{s\vee}_{\mathbf{b}}$.

The maps $\psi,\ \psi^\vee : \Omega_1 \longrightarrow \mathbf{b}$ can be extended to $Y^s_{\mathbf{b}}$ and $Y^{s\vee}_{\mathbf{b}}$, respectively. We will denote these maps by $\psi_{\mathbf{b}}$ and $\psi^\vee_{\mathbf{b}}$. Then $\psi_{\mathbf{b}} : Y^s_{\mathbf{b}} \longrightarrow \mathbf{b}$ denotes the map defined by $\psi_{\mathbf{b}}(x) := \psi((x_1, x_2))$ and $\psi^\vee_{\mathbf{b}} : Y^{s\vee}_{\mathbf{b}} \longrightarrow \mathbf{b}$ denotes the map defined by $\psi^\vee_{\mathbf{b}}(z) := \psi^\vee((z_1, z_2))$.

6.6. Analytical subspaces of $Y^s_{\mathbf{b}}$ and $Y^{s\vee}_{\mathbf{b}}$ for vertices and edges in b. For each vertex $\mathbf{v} \in \mathbf{b}$ we define an analytical space $\Sigma^\sharp_{\mathbf{v}}$. The definition of the space depends on the type $\tau(\mathbf{v})$ of the vertex. If $\tau(\mathbf{v}) = 0$, then we define $\Sigma^\sharp_{\mathbf{v}}$ by $\Sigma^\sharp_{\mathbf{v}} := \{x \in Y^s_{\mathbf{b}} \mid d_{\mathbf{b}}(\psi_{\mathbf{b}}(x), \mathbf{v}) < 1,\ x_0^{q+1} + F_{\mathbf{b}}((x_1, x_2)) = 0\}$. For a vertex $\mathbf{v}$ of type $\tau(\mathbf{v}) = 1$, we define $\Sigma^\sharp_{\mathbf{v}}$ by $\Sigma^\sharp_{\mathbf{v}} := \{z \in Y^{s\vee}_{\mathbf{b}} \mid d_{\mathbf{b}}(\psi^\vee_{\mathbf{b}}(z), \mathbf{v}) < 1,\ -\pi \cdot z_0^{q+1} + F^\vee_{\mathbf{b}}((z_1, z_2)) = 0\}$.

For an edge $\mathbf{e} \in \mathbf{b}$ we define two spaces $\Sigma^\sharp_{\mathbf{e}}$ and $\Sigma^{\sharp\vee}_{\mathbf{e}}$. Let $\Sigma^\sharp_{\mathbf{e}} := \{x \in \Sigma^\sharp_{\mathbf{v}} \mid \psi_{\mathbf{b}}(x) \in \mathbf{e},\ \psi_{\mathbf{b}}(x) \neq \mathbf{v}\} \subset Y^s_{\mathbf{b}}$, where $\mathbf{v} \in \mathbf{e}$ is the vertex of type

$\tau(\mathbf{v}) = 0$. Let $\Sigma_{\mathbf{e}}^{\sharp\vee} := \{z \in \Sigma_{\mathbf{v}'}^{\sharp} \mid \psi_{\mathbf{b}}^{\vee}(z) \in \mathbf{e},\ \psi_{\mathbf{b}}^{\vee}(z) \neq \mathbf{v}'\} \subset Y_{\mathbf{b}}^{s\vee}$, where $\mathbf{v}' \in \mathbf{e}$ is the vertex of type $\tau(\mathbf{v}') = 1$.

6.7 Proposition. *Let $\mathbf{v} \in \mathbf{b}$ be a vertex. Then $\Sigma_{\mathbf{v}}^{\sharp} \cong \Sigma_{\mathbf{v}}$.*

Proof. It is sufficient to prove the proposition for the vertices $\mathbf{v}_0$ and $\mathbf{v}_1$ of type $\tau(\mathbf{v}_i) = i$, $i = 0, 1$ that are contained in the standard edge $\mathbf{e}_0 \in \mathbf{b}$. We have to show that $\Sigma_{\mathbf{v}_0}^{\sharp} \cong \Sigma_{\mathbf{v}_0}$ and that $\Sigma_{\mathbf{v}_1}^{\sharp} \cong \Sigma_{\mathbf{v}_1}$. For this it is sufficient to show that $\{x \in \Sigma_{\mathbf{v}_0}^{\sharp} \mid \psi_{\mathbf{b}}(x) \in \mathbf{e}_0\} \cong \{x \in \Sigma_{\mathbf{v}_0} \mid \psi(\varphi(x)) \in \mathbf{e}_0\}$ and that $\{z \in \Sigma_{\mathbf{v}_1}^{\sharp} \mid \psi_{\mathbf{b}}^{\vee}(z) \in \mathbf{e}_0\} \cong \{z \in \Sigma_{\mathbf{v}_1} \mid \psi^{\vee}(\varphi(z)) \in \mathbf{e}_0\}$. Therefore we have to show that $\{x \in \Sigma_{\mathbf{v}_0}^{\sharp} \mid \psi_{\mathbf{b}}(x) \in \mathbf{e}_0\} \cong X_{\mathbf{e}_0}^{\Sigma} \bigcap \Sigma_{\mathbf{v}_0}$ and $\{z \in \Sigma_{\mathbf{v}_1}^{\sharp} \mid \psi_{\mathbf{b}}^{\vee}(z) \in \mathbf{e}_0\} \cong X_{\mathbf{e}_0}^{\Sigma} \bigcap \Sigma_{\mathbf{v}_1}$.

To prove statement for vertices $\mathbf{v}$ of type $\tau(\mathbf{v}) = 0$, we embed $X_{\mathbf{e}_0}^{\Sigma}$ into $Y_{\mathbf{b}}^{s} \subset \mathbb{P}$ as the set of points such that $\psi((x_1, x_2)) \in \mathbf{e}_0$ and $(\frac{x_0}{x_2})^{q+1} = -f_{\mathbf{e}_0}((x_1, x_2))$. We consider $X_{\mathbf{e}_0}^{\Sigma} \bigcap \Sigma_{\mathbf{v}_0}$ therefore as being the subset $\{x \in X_{\mathbf{e}_0}^{\Sigma} \mid d_{\mathbf{b}}(\psi((x_1, x_2)), \mathbf{v}_0) < 1\} = \{x \in X_{\mathbf{e}_0}^{\Sigma} \mid |\frac{\pi x_2}{x_1}| < 1\}$ of $Y_{\mathbf{b}}^{s}$.

Since $|\frac{\pi x_2}{x_1}| < 1$, one has that $f_{\mathbf{e}_0}((x_1, x_2)) = \frac{x_1}{x_2} \cdot \frac{1 + (\frac{x_1}{x_2})^{(q-1)}}{1 + (\frac{-\pi \cdot x_2}{x_1})^{(q-1)}} \equiv \frac{x_1}{x_2} + (\frac{x_1}{x_2})^{q}$ mod π. Therefore $x_0^{q+1} = -x_2^{q+1} f_{\mathbf{e}_0}((x_1, x_2)) \equiv -x_1^{q} x_2 - x_1 x_2^{q} \mod \pi$ holds. Since $x_1^{q} x_2 + x_1 x_2^{q} \equiv F_{\mathbf{b}}((x_1, x_2)) \mod \pi$, it follows that $X_{\mathbf{e}_0}^{\Sigma} \bigcap \Sigma_{\mathbf{v}_0} \cong \{x \in Y_{\mathbf{b}}^{s} \mid \psi_{\mathbf{b}}(x) \in \mathbf{e}_0, d_{\mathbf{b}}(\psi_{\mathbf{b}}(x), \mathbf{v}_0) < 1,\ x_0^{q+1} + F_{\mathbf{b}}((x_1, x_2)) = 0\} = \{x \in \Sigma_{\mathbf{v}_0}^{\sharp} \mid \psi_{\mathbf{b}}(x) \in \mathbf{e}_0\}$. This proves the proposition for the vertices $\mathbf{v}$ of type $\tau(\mathbf{v}) = 0$.

The statement of the proposition is proved similarly for the vertex $\mathbf{v}_1$ of type $\tau(\mathbf{v}_1) = 1$. One embeds $X_{\mathbf{e}_0}^{\Sigma}$ into $Y_{\mathbf{b}}^{s\vee} \subset \mathbb{P}^{\vee}$ as the set of points such that $\psi^{\vee}((z_1, z_2)) \in \mathbf{e}_0$ and $(\frac{z_0}{z_1})^{q+1} = -1/f_{\mathbf{e}_0}((z_1, z_2))$. Since $|\frac{z_1}{z_2}| < 1$ on $X_{\mathbf{e}_0}^{\Sigma} \bigcap \Sigma_{\mathbf{v}_1}^{\sharp}$, it follows that $-\pi \cdot z_0^{q+1} \equiv -(z_1^{q} \pi z_2 + z_1 (\pi z_2)^{q}) \equiv -F_{\mathbf{b}}^{\vee}((z_1, z_2))$ mod π holds. From this the proposition follows. □

6.8 Proposition. *Let $g \in SU(2, L)$ and let $\mathbf{e} = g(\mathbf{e}_0) \in \mathbf{b}$ be an edge. Then the spaces $\Sigma_{\mathbf{e}}^{\sharp}$ and $\Sigma_{\mathbf{e}}^{\sharp\vee}$ are isomorphic.*

The isomorphism $\Sigma_{\mathbf{e}}^{\sharp} \cong \Sigma_{\mathbf{e}}^{\sharp\vee}$ is given by taking $z_1 = x_1$, $z_2 = -x_2$ and as z_0 the unique solution of $z_0^{q+1} = \frac{F_{\mathbf{b}}((x_1, x_2)) F_{\mathbf{b}}^{\vee}((z_1, z_2))}{-\pi \cdot x_0^{q+1}}$ that satisfies $z_0 \equiv \frac{g^* x_1 g^* x_2}{x_0}$ mod π. The isomorphism does not depend on the choice of the element $g \in SU(2, L)$ such that $g(\mathbf{e}_0) = \mathbf{e}$.

Proof. The action of the group $SU(2, L)$ on $\mathbb{P}$ and $\mathbb{P}^{\vee}$ preserves the quadratic form $x_1 z_2 + x_2 z_1$. Therefore identifying $z_1 = x_1$ and $z_2 = -x_2$ on $\Sigma_{\mathbf{e}}^{\sharp}$ and $\Sigma_{\mathbf{e}}^{\sharp\vee}$ implies that $g^* z_1 = g^* x_1$ and $g^* z_2 = -g^* x_2$ holds. Then $1 < |\frac{g^* x_1}{g^* x_2}| = |\frac{g^* z_1}{g^* z_2}| < |\pi|$.

Once one has identified $\Sigma_{\mathbf{e}}^{\sharp}$ with $\Sigma_{\mathbf{e}}^{\sharp\vee}$, the equation $z_0^{q+1} = \frac{F_{\mathbf{b}}(x)F_{\mathbf{b}}^{\vee}(z)}{-\pi\cdot x_0^{q+1}}$ holds. We use this equation to derive a relation between the coordinates x_0 and z_0 that can be used to obtain the isomorphism between the two spaces. Since $1 < |\frac{g^*x_1}{g^*x_2}| < |\pi|$, we have $F_{\mathbf{b}}((x_1,x_2))F_{\mathbf{b}}^{\vee}((z_1,z_2))/(-\pi) \equiv a_{\mathbf{v},\mathbf{b}}((x_1,x_2))a_{\mathbf{v}'}((z_1,z_2))/(-\pi) \equiv (g^*x_1g^*x_2^q+g^*x_2g^*x_1^q)(g^*x_1(-\pi g^*x_2)^q+(-\pi g^*x_2)g^*x_1^q)/(-\pi) \equiv (g^*x_1g^*x_2)^{q+1} \mod \pi$.

Therefore we can identify the coordinate z_0 with the unique solution of the equation above such that $z_0 \equiv \frac{g^*x_1g^*x_2}{x_0} \mod \pi$ as stated in the proposition.

This defines a bijection between the coordinates x_i, $i = 0,1,2$ on $\Sigma_{\mathbf{e}}^{\sharp}$ and the coordinates z_i, $i = 0,1,2$ on $\Sigma_{\mathbf{e}}^{\sharp\vee}$. Hence this gives an isomorphism between $\Sigma_{\mathbf{e}}^{\sharp}$ and $\Sigma_{\mathbf{e}}^{\sharp\vee}$.

If $g,g' \in SU(2,L)$ are elements such that $g(\mathbf{e}_0) = g'(\mathbf{e}_0) = \mathbf{e}$, then $|\frac{g'^*x_i}{g^*x_i}| \equiv 1 \mod \pi$ for $i = 1,2$. Therefore the choice of z_0 does not depend on the element $g \in SU(2,L)$ used in the proposition. Hence the identification $\Sigma_{\mathbf{e}}^{\sharp} \cong \Sigma_{\mathbf{e}}^{\sharp\vee}$ does not depend on the choice of the element $g \in SU(2,L)$. □

6.9. Identifying coordinates in $\mathbb{P}$ and $\mathbb{P}^{\vee}$. The identifications of the coordinates x_i and z_i, $i = 1,2$ using the equation $x_1z_2 + x_2z_2 = 0$ can easily be done in advance for all of $\mathbb{P}$ and $\mathbb{P}^{\vee}$. Then $\mathbb{P}$ and $\mathbb{P}^{\vee}$ have the line $(0,x_1,x_2) \times (0,z_1,z_2)$ with the equation $x_1z_2 + x_2z_2 = 0$ in common. This line is preserved by the action of the group $SU(2,L)$.

The resulting variety contains the blow up of $\mathbb{P}$ in the point $(x_0,0,0)$ and the blow up of $\mathbb{P}^{\vee}$ in $(z_0,0,0)$. These are the points that are fixed under the action of the group $SU(2,L)$.

The exceptional line $x_1 = x_2 = 0$ of the blow up of $\mathbb{P}$ is identified with the ordinary line $(0,z_1,z_2) \subset \mathbb{P}^{\vee}$ and the exceptional line $z_1 = z_2 = 0$ of the blow up of $\mathbb{P}^{\vee}$ is identified with the ordinary line $(0,x_1,x_2) \subset \mathbb{P}$ by the equation $x_1z_2 + x_2z_2 = 0$. Since $(x_0,0,0) \notin Y_{\mathbf{b}}^s$ and $(z_0,0,0) \notin Y_{\mathbf{b}}^{s\vee}$, the spaces $Y_{\mathbf{b}}^s \subset \mathbb{P}$ and $Y_{\mathbf{b}}^{s\vee} \subset \mathbb{P}^{\vee}$ are not affected by the blow ups.

6.10 Proposition. *Let $\mathbf{e} \in \mathbf{b}$ be an edge and let $\mathbf{v},\mathbf{v}' \in \mathbf{e}$ be the vertices of type $\tau(\mathbf{v}) = 0$ and $\tau(\mathbf{v}') = 1$. The analytical spaces $\Sigma_{\mathbf{v}}^{\sharp} \subset Y_{\mathbf{b}}^s$ and $\Sigma_{\mathbf{v}'}^{\sharp} \subset Y_{\mathbf{b}}^{s\vee}$ can be glued together by identifying the open admissable subspaces $\Sigma_{\mathbf{e}}^{\sharp} \subset \Sigma_{\mathbf{v}}^{\sharp}$ and $\Sigma_{\mathbf{e}}^{\sharp\vee} \subset \Sigma_{\mathbf{v}'}^{\sharp}$. Then the image of $\{x \in \Sigma_{\mathbf{v}}^{\sharp} \mid \psi_{\mathbf{b}}(x) \in \mathbf{e}\} \bigcup \{z \in \Sigma_{\mathbf{v}'}^{\sharp} \mid \psi_{\mathbf{b}}^{\vee}(z) \in \mathbf{e}\}$ in the resulting space is an affinoid isomorphic to $X_{\mathbf{e}}^{\Sigma}$.*

Proof. Let us define $\Sigma_{\mathbf{v}}^{\sharp}(\mathbf{e}) := \{x \in \Sigma_{\mathbf{v}}^{\sharp} \mid \psi_{\mathbf{b}}(x) \in \mathbf{e}\} = \{x \in Y_{\mathbf{b}}^s \mid \psi_{\mathbf{b}}(x) \in \mathbf{e}, \psi_{\mathbf{b}}(x) \neq \mathbf{v}', x_0^{q+1} = -F_{\mathbf{b}}((x_1,x_2))\} \subset \Sigma_{\mathbf{v}}^{\sharp} \subset Y_{\mathbf{b}}^s$. Similarly, we define $\Sigma_{\mathbf{v}'}^{\sharp}(\mathbf{e})$

as $\Sigma^\sharp_{\mathbf{v}'}(\mathbf{e}) := \{z \in \Sigma^\sharp_{\mathbf{v}'} \mid \psi^\vee_{\mathbf{b}}(z) \in \mathbf{e}\} = \{z \in Y^{s\vee}_{\mathbf{b}} \mid \psi^\vee_{\mathbf{b}}(z) \in \mathbf{e}, \psi^\vee_{\mathbf{b}}(z) \neq \mathbf{v}, \pi \cdot z_0^{q+1} = F^\vee_{\mathbf{b}}((z_1, z_2))\} \subset \Sigma^\sharp_{\mathbf{v}'} \subset Y^{s\vee}_{\mathbf{b}}$. Then $\Sigma^\sharp_{\mathbf{e}} \subset \Sigma^\sharp_{\mathbf{v}}(\mathbf{e})$ and $\Sigma^{\sharp\vee}_{\mathbf{e}} \subset \Sigma^\sharp_{\mathbf{v}'}(\mathbf{e})$. In fact $\Sigma^\sharp_{\mathbf{e}} = \{x \in \Sigma^\sharp_{\mathbf{v}}(\mathbf{e}) \mid 1 < |\frac{g^* x_1}{g^* x_2}| < |\pi|\}$ and $\Sigma^{\sharp\vee}_{\mathbf{e}} = \{z \in \Sigma^\sharp_{\mathbf{v}'}(\mathbf{e}) \mid 1 < |\frac{g^* z_1}{g^* z_2}| < |\pi|\}$.

Let us glue $\Sigma^\sharp_{\mathbf{v}}(\mathbf{e})$ and $\Sigma^\sharp_{\mathbf{v}'}(\mathbf{e})$ by using the isomorphism between $\Sigma^\sharp_{\mathbf{e}}$ and $\Sigma^{\sharp\vee}_{\mathbf{e}}$. In the proof above that $\Sigma_{\mathbf{v}} \cong \Sigma^\sharp_{\mathbf{v}}$, we have already shown that $\Sigma^\sharp_{\mathbf{v}}(\mathbf{e}) \cong \Sigma_{\mathbf{v}} \bigcap X^\Sigma_{\mathbf{e}}$ and that $\Sigma^\sharp_{\mathbf{v}'}(\mathbf{e}) \cong \Sigma_{\mathbf{v}'} \bigcap X^\Sigma_{\mathbf{e}}$. On $\Sigma^\sharp_{\mathbf{e}} \cong \Sigma^{\sharp\vee}_{\mathbf{e}}$ the equation $(\frac{x_0}{z_0})^{q+1}/(-\pi) = \frac{F_{\mathbf{b}}(x)}{F^\vee_{\mathbf{b}}(z)}$ holds. Multiplying the left side by $\pi \cdot z_0^{q+1}$ and the right side by $F^\vee_{\mathbf{b}}(z)$ one obtains the equation $x_0^{q+1} + F_{\mathbf{b}}(x) = 0$ that defines the space $\Sigma^\sharp_{\mathbf{v}}$. Similarly, one can use the equation $-\pi \cdot (\frac{z_0}{x_0})^{q+1} = \frac{F^\vee_{\mathbf{b}}(z)}{F_{\mathbf{b}}(x)}$ and multiply the left side by x_0^{q+1} and the right side by $F_{\mathbf{b}}(x)$ to obtain the equation that defines the space $\Sigma^\sharp_{\mathbf{v}'}$. Therefore the space constructed is indeed isomorphic to the affinoid space $X^\Sigma_{\mathbf{e}}$.

The covering $\{\Sigma^\sharp_{\mathbf{v}}(\mathbf{e}), \Sigma^\sharp_{\mathbf{e}} \cong \Sigma^{\sharp\vee}_{\mathbf{e}}, \Sigma^\sharp_{\mathbf{v}'}(\mathbf{e})\}$ is an open admissable covering of the affinoid space isomorphic to $X^\Sigma_{\mathbf{e}} \subset \Sigma$. The covering $\{\Sigma^\sharp_{\mathbf{v}}, \Sigma^\sharp_{\mathbf{e}} \cong \Sigma^{\sharp\vee}_{\mathbf{e}}, \Sigma^\sharp_{\mathbf{v}'}\}$ is an open admissable covering of the space obtained by glueing $\Sigma^\sharp_{\mathbf{v}}$ and $\Sigma^\sharp_{\mathbf{v}'}$ along $\Sigma^\sharp_{\mathbf{e}} \cong \Sigma^{\sharp\vee}_{\mathbf{e}}$. □

6.11 Theorem. *Let $\Sigma^\sharp := \bigcup_{\mathbf{v}\in\mathbf{b}} \Sigma^\sharp_{\mathbf{v}}/\sim$. Here $\sim$ denotes the equivalence relation obtained by applying the isomorphisms $\Sigma^\sharp_{\mathbf{e}} \cong \Sigma^{\sharp\vee}_{\mathbf{e}}$ for all edges $\mathbf{e} \in \mathbf{b}$. Then $\Sigma^\sharp$ is a well-defined rigid analytic variety and $\Sigma^\sharp \cong \Sigma$.*

Proof. In the proposition above it is proved that for the vertices $\mathbf{v}, \mathbf{v}' \in \mathbf{e}$ the spaces $\Sigma^\sharp_{\mathbf{v}}$ and $\Sigma^\sharp_{\mathbf{v}'}$ can be glued by applying the isomorphism $\Sigma^\sharp_{\mathbf{e}} \cong \Sigma^{\sharp\vee}_{\mathbf{e}}$. Since the spaces $\Sigma^\sharp_{\mathbf{e}}$ are disjoint for the edges $\mathbf{e} \in \mathbf{b}$, one can use all the identifications $\Sigma^\sharp_{\mathbf{e}} \cong \Sigma^{\sharp\vee}_{\mathbf{e}}$ simultaneously to obtain a well-defined rigid analytic variety $\Sigma^\sharp = \bigcup_{\mathbf{v}\in\mathbf{b}} \Sigma^\sharp_{\mathbf{v}}/\sim$. Since this space consists of an affinoid isomorphic to $X^\Sigma_{\mathbf{e}}$ for each edge $\mathbf{e}$ in $\mathbf{b}$, the space $\Sigma^\sharp$ is isomorphic to Σ. This concludes the proof of the theorem. □

6.12 Corollary. *The following statements hold:*

i) *Γ acts linearly on $\bigcup_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=0} \Sigma^\sharp_{\mathbf{v}} \subset Y^s_{\mathbf{b}}$ through the coordinates x_i, $i = 0, 1, 2$.*

ii) *Γ acts linearly on $\bigcup_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=1} \Sigma^\sharp_{\mathbf{v}} \subset Y^{s\vee}_{\mathbf{b}}$ through the coordinates z_i, $i = 0, 1, 2$.*

iii) We have a map $\psi_{\mathbf{b}}^{\Sigma^\sharp} : \Sigma^\sharp \to \mathbf{b}$, such that $\psi_{\mathbf{b}}^{\Sigma^\sharp}(x) = \psi_{\mathbf{b}}(x)$ for $x \in \Sigma_{\mathbf{v}}^\sharp$ with $\tau(\mathbf{v}) = 0$ and $\psi_{\mathbf{b}}^{\Sigma^\sharp}(z) = \psi_{\mathbf{b}}^\vee(z)$ for $z \in \Sigma_{\mathbf{v}}^\sharp$ with $\tau(\mathbf{v}) = 1$.

Proof. The linear action of Γ follows directly from the construction of the spaces and the fact that the infinite sums $F_{\mathbf{b}}((x_1, x_2))$ and $F_{\mathbf{b}}((z_1, z_2))$ are Γ-invariant.

Statement (iii) of the corollary follows directly from the fact that the maps $\psi_{\mathbf{b}}$ and $\psi_{\mathbf{b}}^\vee$ coincide on $\Sigma_{\mathbf{e}}^\sharp$ and $\Sigma_{\mathbf{e}}^{\sharp\vee}$ for $\mathbf{e} \in \mathbf{b}$. □

6.13 Remark. The glueing done above can also be done by defining the coordinate z_0 using the Γ-invariant infinite sum $Z_{\mathbf{b}}((x_1, x_2))$ for all edges $\mathbf{e} \in \mathbf{b}$. To give the isomorphism between $\Sigma_{\mathbf{e}}^\sharp$ and $\Sigma_{\mathbf{e}}^{\sharp\vee}$ one takes the solution of $z_0^{q+1} = \frac{F_{\mathbf{b}}((x_1,x_2))F_{\mathbf{b}}^\vee((z_1,z_2))}{-\pi \cdot x_0^{q+1}}$ such that $z_0 \equiv \frac{Z_{\mathbf{b}}((x_1,x_2))}{x_0} \mod \pi$ holds.

7 Another equivariant embedding into the projective plane

In this section we give a different embedding of the admissable subspaces $\Sigma_{\mathbf{v}} \subset \Sigma$ into $\mathbb{P} \cong \mathbb{P}_L^2$ for the vertices $\mathbf{v} \in \mathbf{b}$ of type $\tau(\mathbf{v}) = 0$. It does not explicitly use the fact that Σ is a covering of Ω_1. It uses explicitly the fact that the component of the reduction of Σ belonging to the vertex $\mathbf{v} \in \mathbf{b}$ is a hermitian curve. This embedding will later be used to construct a space $\mathcal{Y}$ on which discrete subgroups of $SU(3, L)$ act with proper quotients.

7.1. Polynomials for the vertices $\mathbf{v} \in \mathbf{b}$ of type $\tau(\mathbf{v}) = 0$. For each vertex $\mathbf{v} \in \mathbf{b}$ of type $\tau(\mathbf{v}) = 0$, we define a homogeneous polynomial $b_{\mathbf{v}}(x)$ of degree $q + 1$ for $x \in Y_{\mathbf{b}}^s$, such that $b_{\mathbf{v}}(x) \equiv x_0^{q+1} + a_{\mathbf{v},\mathbf{b}}((x_1, x_2)) \mod \pi$. The polynomials satisfy the following condition:

For all $\gamma \in \Gamma$ and all $\mathbf{v} \in \mathbf{b}$ of type $\tau(\mathbf{v}) = 0$ one has $b_{\gamma(\mathbf{v})}(x) = \gamma^* b_{\mathbf{v}}(x)$.

From now on we change the notation a little. We view the polynomials $a_{\mathbf{v},\mathbf{b}}(z)$ and function $F_{\mathbf{b}}^\vee(z)$ as defined for points $z \in Y_{\mathbf{b}}^{s\vee}$ and the function $F_{\mathbf{b}}(x)$ as defined for points $x \in Y_{\mathbf{b}}^s$.

7.2 Lemma. *Let $\mathbf{v} \in \mathbf{b}$ be a vertex of type $\tau(\mathbf{v}) = 0$ and let $\mathbf{e} \ni \mathbf{v}$ be an edge. Let $A \in \mathbf{b}$ be an apartment that contains the edge $\mathbf{e}$. Let $x \in \Sigma_{\mathbf{v}}^\sharp \subset Y_{\mathbf{b}}^s$ be a point such that $\psi_{\mathbf{b}}(x) \in \mathbf{e}$. Let x_0, x_1, x_2 be the coordinates belonging to the apartment A. Then $v(\frac{x_1 x_2}{x_0^2}) = \frac{q-1}{q+1} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi_{\mathbf{b}}(x))$ and $1 \geq |\frac{x_1 x_2}{x_0^2}| >$*

$|\pi|^{(q-1)/(q+1)}$. *Moreover,* $|\frac{x_1x_2}{x_0^2}| = 1$ *if and only if* $\psi_{\mathbf{b}}(x)$ *is the vertex* $\mathbf{v}$ *of type* $\tau(\mathbf{v}) = 0$

Proof. Let $\mathbf{v} \in A$ be the vertex of type $\tau(\mathbf{v}) = 0$ closest to $\psi_{\mathbf{b}}(x)$. Then $F_{\mathbf{b}}((x_1, x_2)) \equiv a_{\mathbf{v},\mathbf{b}}((x_1, x_2)) \mod \pi$ and $|x_0^{q+1}| = |a_{\mathbf{v},\mathbf{b}}((x_1, x_2))|$. Hence $v(\frac{x_0^{q+1}}{(x_1x_2)^{(q+1)/2}}) = v(\frac{a_{\mathbf{v},\mathbf{b}}((x_1,x_2))}{(x_1x_2)^{(q+1)/2}}) = -\frac{q-1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi_{\mathbf{b}}(x))$ by one of the lemmas above. Therefore $v(\frac{x_0^2}{(x_1x_2)}) = -\frac{q-1}{q+1} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi_{\mathbf{b}}(x))$. Since $0 \leq d_{\mathbf{b}}(\mathbf{v}, \psi_{\mathbf{b}}(x)) < 1$ the second statement follows. Moreover, if and only if $d_{\mathbf{b}}(\mathbf{v}, \psi_{\mathbf{b}}(x)) = 0$ does $|\frac{x_1x_2}{x_0^2}| = 1$ hold. □

7.3. Analytical subspaces of $Y_{\mathbf{b}}^s$ and $Y_{\mathbf{b}}^{s\vee}$ for vertices and edges in b. Let $\mathbf{v} \in \mathbf{b}$ be a vertex of type $\tau(\mathbf{v})$. If $\tau(\mathbf{v}) = 0$, then we take $\Sigma_{\mathbf{v},\mathbf{b}} := \{x \in Y_{\mathbf{b}}^s \mid d_{\mathbf{b}}(\psi_{\mathbf{b}}(x), \mathbf{v}) < 1,\ b_{\mathbf{v}}(x) = 0\}$. If $\tau(\mathbf{v}) = 1$, then $\Sigma_{\mathbf{v},\mathbf{b}} := \Sigma_{\mathbf{v}}^\sharp \subset Y_{\mathbf{b}}^{s\vee}$.

For edges $\mathbf{e} \in \mathbf{b}$ we define two analytical spaces $\Sigma_{\mathbf{e},\mathbf{b}} \subset Y_{\mathbf{b}}^s$ and $\Sigma_{\mathbf{e},\mathbf{b}}^\vee \subset Y_{\mathbf{b}}^{s\vee}$. Let $\Sigma_{\mathbf{e},\mathbf{b}} := \{x \in \Sigma_{\mathbf{v},\mathbf{b}} \mid \psi_{\mathbf{b}}(x) \in \mathbf{e}, \psi_{\mathbf{b}}(x) \neq \mathbf{v}\}$, where $\mathbf{v} \in \mathbf{e}$ is the vertex of type $\tau(\mathbf{v}) = 0$. Let $\Sigma_{\mathbf{e},\mathbf{b}}^\vee := \Sigma_{\mathbf{e}}^{\sharp\vee}$.

7.4 Proposition. *Let* $\mathbf{v} \in \mathbf{b}$ *be a vertex of type* $\tau(\mathbf{v}) = 0$*. Then* $\Sigma_{\mathbf{v},\mathbf{b}} \cong \Sigma_{\mathbf{v}}^\sharp$*.*

Proof. The space $\Sigma_{\mathbf{v}}^\sharp$ is defined as $\Sigma_{\mathbf{v}}^\sharp = \{x \in Y_{\mathbf{b}}^s \mid d_{\mathbf{b}}(\psi_{\mathbf{b}}(x), \mathbf{v}) < 1, x_0^{q+1} = -F_{\mathbf{b}}((x_1, x_2))\}$. Furthermore, $F_{\mathbf{b}}((x_1, x_2)) \equiv a_{\mathbf{v},\mathbf{b}}((x_1, x_2)) \mod \pi$. Therefore $x_0^{q+1} + a_{\mathbf{v},\mathbf{b}}((x_1, x_2)) \cdot (F_{\mathbf{b}}((x_1, x_2))/a_{\mathbf{v},\mathbf{b}}((x_1, x_2))) = 0$ holds with $F_{\mathbf{b}}((x_1, x_2))/a_{\mathbf{v},\mathbf{b}}((x_1, x_2)) \equiv 1 \mod \pi$. For $x \in \Sigma_{\mathbf{v},\mathbf{b}}$, one has that $b_{\mathbf{v}}(x) \equiv x_0^{q+1} + a_{\mathbf{v},\mathbf{b}}((x_1, x_2)) \equiv 0 \mod \pi$ holds. From this the isomorphism follows. □

7.5 Proposition. *Let* $g \in SU(2, L)$ *and let* $\mathbf{e} = g(\mathbf{e}_0) \in \mathbf{b}$ *be an edge. Then the spaces* $\Sigma_{\mathbf{e},\mathbf{b}}$ *and* $\Sigma_{\mathbf{e},\mathbf{b}}^\vee$ *are isomorphic.*

The isomorphism $\Sigma_{\mathbf{e},\mathbf{b}} \cong \Sigma_{\mathbf{e},\mathbf{b}}^\vee$ *is given by taking* $z_1 = x_1$*,* $z_2 = -x_2$ *and as* z_0 *the solution of* $z_0^{q+1} = \frac{a_{\mathbf{v},\mathbf{b}}((x_1,x_2))F_{\mathbf{b}}^\vee((z_1,z_2))}{-\pi \cdot x_0^{q+1}}$ *that satisfies* $z_0 \equiv \frac{g^*x_1 g^*x_2}{x_0} \mod \pi$*. The identifications given do not depend on the choice of the element* $g \in SU(2, L)$ *such that* $g(\mathbf{e}_0) = \mathbf{e}$*.*

Proof. Similar to the proof that $\Sigma_{\mathbf{e}}^\sharp$ and $\Sigma_{\mathbf{e}}^{\sharp\vee}$ are isomorphic, once one observes that $a_{\mathbf{v},\mathbf{b}}((x_1, x_2)) \equiv F_{\mathbf{b}}((x_1, x_2)) \mod \pi$ on $\Sigma_{\mathbf{v},\mathbf{b}} \supset \Sigma_{\mathbf{e},\mathbf{b}}$. □

7.6 Theorem. *Let* $\Sigma_{\mathbf{b}} := \bigcup_{\mathbf{v}\in\mathbf{b}} \Sigma_{\mathbf{v},\mathbf{b}}/\sim$*. Here* $\sim$ *denotes the equivalence relation obtained by applying the isomorphisms* $\Sigma_{\mathbf{e},\mathbf{b}} \cong \Sigma_{\mathbf{e},\mathbf{b}}^\vee$ *for all edges* $\mathbf{e} \in \mathbf{b}$*. Then* $\Sigma_{\mathbf{b}}$ *is a well-defined rigid analytic variety and* $\Sigma_{\mathbf{b}} \cong \Sigma$*.*

Proof. Similar to the proof that $\Sigma^\sharp$ is a well-defined rigid analytic variety isomorphic to Σ. $\square$

7.7. Other possible simplifications. One can again simplify the construction somewhat by identifying the coordinates x_1, x_2 of $\mathbb{P}$ with the coordinates z_1, z_2 of $\mathbb{P}^\vee$ through the relation $x_1 z_2 + x_2 z_1 = 0$.

A more significant simplification can be obtained by also defining polynomials $b_{\mathbf{v}}(z) \equiv -\pi \cdot z_0^{q+1} + a_{\mathbf{v},\mathbf{b}}(z) \mod \pi$ for the vertices $\mathbf{v} \in \mathbf{b}$ of type $\tau(\mathbf{v}) = 1$. These polynomials can then be used to define for vertices $\mathbf{v} \in \mathbf{b}$ of type $\tau(\mathbf{v}) = 1$ a analytical space $\Sigma'_{\mathbf{v},\mathbf{b}} := \{z \in Y_{\mathbf{b}}^{s\vee} \mid b_{\mathbf{v}}(z) = 0, d_{\mathbf{b}}(\psi_{\mathbf{b}}^\vee(z), \mathbf{v}) < 1\}$. The proofs in this case are again quite similar to the ones presented above. For our purposes in the sections below the construction of $\Sigma_{\mathbf{b}}$ as presented here is sufficient.

8 Stable points in the projective plane

In this section we describe in some detail the points in the projective plane that are stable for all maximal K-split tori of the group $SU(3, L)$. In particular, we describe a $SU(3, L)$-equivariant map from the set of stable points to the building $\mathcal{B}$. The results presented here have been proved in [LV] and [V]. They will be used to construct the uniformising space in the sections that follow.

8.1. Stable and semistable points. Let $\mathbb{P}$ and $\mathbb{P}^\vee$ be distinct projective planes $\mathbb{P}^2_L$ with coordinates x_0, x_1, x_2 and z_0, z_1, z_2, respectively. Let the group $SU(3, L)$ act linearly on $\mathbb{P}$ and $\mathbb{P}^\vee$ preserving the hermitian form h and $h^\vee$, respectively.

For a maximal K-split torus in $SU(3, L)$ we use the linearisation that is the restriction to the torus of the $SU(3, L)$-linearisation of $\mathcal{O}(1)$. The homogeneous torus invariants are generated by the monomials x_0^2 and $x_1 x_2$ in $\mathbb{P}$ and by the monomials z_0^2 and $z_1 z_2$ in $\mathbb{P}^\vee$

Let $Y^{ss} \subset \mathbb{P}$ and $Y^{ss\vee} \subset \mathbb{P}^\vee$ denote the open analytical subspaces that contain the points that are semistable for all maximal K-split tori in $SU(3, L)$. Similarly, we denote by $Y^s \subset \mathbb{P}$ and $Y^{s\vee} \subset \mathbb{P}^\vee$ the open analytical subspaces consisting of the points that are stable for all maximal K-split tori in $SU(3, L)$. Then $Y^{ss} := \mathbb{P}^2_L - \{y \in \mathbb{P}^2(L) \mid h(y, y) = 0\}$ and $Y^s := \{x \in \mathbb{P}^2_L \mid \forall (y \in \mathbb{P}^2(L) \text{ such that } h(y, y) = 0)\ h(x, y) \neq 0\}$. Similar descriptions hold for $Y^{ss\vee}$ and $Y^{s\vee}$.

8.2. Criterion for semistability. We define a function $r(x)$ on $\mathbb{P}^2_L$ involving torus invariants that can be used to define the space Y^{ss} .

Let $A \subset \mathcal{B}$ be an apartment with coordinates x_0, x_1, x_2 and let $g \in SU(3, L)$. Then we define:

$$r_{gA,A}(x) := \begin{cases} 0 & \text{if } x_0^2 = x_1x_2 = 0 \\ max\{|g^*x_0^2|, |g^*x_1g^*x_2|\}/max\{|x_0^2|, |x_1x_2|\} & \text{if } max\{|x_0^2|, |x_1x_2|\} \neq 0. \end{cases}$$

Then $r_{gA,A}(x)$ is well-defined for $x \in \mathbb{P}^2_L$.

Let $r(x) := \inf\{r_{gA,A}(x) \mid g \in SU(3, L)\}$ for $x \in \mathbb{P}^2_L$. Then $r(x) > 0$ if and only if $x \in Y^{ss}$ and there exists an apartment $gA \in \mathcal{B}$ such that $r_{gA,A}(x) = r(x)$ (See [PV] §3.6).

8.3. The interval of semistability. Let $\mathbf{v} \in \mathcal{B}$ be a vertex of type $\tau(\mathbf{v}) = 0$. To the vertex $\mathbf{v}$ belongs an equivalence class of L°-modules $[M_\mathbf{v}]$. Let $M_\mathbf{v}$ be the module $< e_0, e_1, e_2 >$.

Let $\mathbb{C}_p^\circ$ be the ring of integers of the completion of the algebraic closure of K. Let $\mathcal{M}_\mathbf{v}$ be the $\mathbb{C}_p^\circ$-module $\mathcal{M}_\mathbf{v} := M_\mathbf{v} \otimes \mathbb{C}_p^\circ$. For a rational point $u \in \mathcal{B}$, there exists an apartment $A \ni \mathbf{v}, u$ and a torus element $s \in S$, where S is the torus belonging to the apartment A, such that $u = s \cdot \mathbf{v}$. To the point $u = s \cdot \mathbf{v} \in \mathcal{B}$ we associate the $\mathbb{C}_p^\circ$-module $\mathcal{M}_u := s \cdot \mathcal{M}_\mathbf{v}$.

The parahoric group $P_u \subset SU(3, L)$ acts on $\mathcal{M}_u$. The equivalence class $[\mathcal{M}_u]$ of $\mathbb{C}_p^\circ$-modules does not depend on the choice of the apartment A, the torus element s or the vertex $\mathbf{v}$.

If for all apartments $A \ni u$ the reduction $\overline{x} \in \mathbb{P}(\mathcal{M}_u \otimes \ell)$ of x is semistable for the reduction $S \otimes \ell$ of the torus S that belongs to the apartment A, then we say that x *is semistable in the reduction for $SU(3, L)$ at $u \in \mathcal{B}$*.

Let $x \in Y^{ss}$ be a point. The *interval of semistability $I(x)$ of x for the group $SU(3, L)$* is the closure in $\mathcal{B}$ of the set of points $u \in \mathcal{B}(\mathbb{Q})$ such that x is semistable in the reduction for the group $SU(3, L)$ at u:
$I(x) := \overline{\{u \in \mathcal{B}(\mathbb{Q}) \mid x \text{ is semistable in the reduction for } SU(3, L) \text{ at } u\}} \subset \mathcal{B}$.

The interval of semistability $I_\mathbf{b}(x)$ of $x \in Y_\mathbf{b}^{ss}$ for the group $SU(2, L)$ acting on the building $\mathbf{b}$ is defined analogously. The point $x \in Y_\mathbf{b}^{ss}$ *is semistable in the reduction for the group $SU(2, L)$ at $u \in \mathbf{b}$*, if for all apartments $A \subset \mathbf{b}$, $A \ni u$, the reduction $\overline{x} \in \mathbb{P}(\mathcal{M}_u \otimes \ell)$ of x is semistable for the reduction $S \otimes \ell$ of the torus S that belongs to the apartment A. Then:
$I_\mathbf{b}(x) := \overline{\{u \in \mathbf{b}(\mathbb{Q}) \mid x \text{ is semistable in the reduction for } SU(2, L) \text{ at } u\}} \subseteq \mathbf{b}$.

The subsets $I_{\mathbf{b}}(x) \subseteq \mathbf{b}$ and $I(x) \subset \mathcal{B}$ are convex for a point $x \in Y_{\mathbf{b}}^{ss}$ and $x \in Y^{ss}$, respectively. The interval $I(x)$ is bounded if and only if $x \in Y^{s}$. (See [V] cor. 4.10.) If $x \in Y^{ss} - Y^{s}$, then the interval of semistability $I(x)$ is not bounded. As an example take the point $(x_0, 0, 0) \in \mathbb{P}_L^2$ that is stabilised by the group $SU(2, L)$ belonging to $\mathbf{b} \subset \mathcal{B}$. Then $I(x) = \mathbf{b}$.

For the action of $SU(2, L)$ on $Y_{\mathbf{b}}^{ss\vee}$ and $SU(3, L)$ on $Y^{ss\vee}$ one analogously defines intervals of semistability $I_{\mathbf{b}}^{\vee}(z)$ and $I^{\vee}(z)$ for points z in $Y_{\mathbf{b}}^{ss\vee}$ and $Y^{ss\vee}$, respectively.

For later use we recall some results from [V]:

8.4 Proposition. *Let $x \in Y^{ss}$ be a point. Then there exists a $SU(2, L)$-building $\mathbf{b} \subset \mathcal{B}$ such that $I(x) = I_{\mathbf{b}}(x)$. In particular, $I(x) \subseteq \mathbf{b}$.*

Proof. See [V] theorem 6.2. □

8.5 Proposition. *Let $x \in Y^{ss}$ and let $\mathbf{b} \subset \mathcal{B}$ be a $SU(2, L)$-building. Then the following two statements hold:*

i) If the intersection $I(x) \bigcap \mathbf{b}$ is non-empty, then $I_{\mathbf{b}}(x) = I(x) \bigcap \mathbf{b}$.

ii) If the intersection $I(x) \bigcap \mathbf{b}$ is empty, then $I_{\mathbf{b}}(x) = \{\mathbf{v}\}$. Here $\mathbf{v} \in \mathbf{b}$ is the unique vertex such that $d_{\mathcal{B}}(\mathbf{v}, I(x)) = d_{\mathcal{B}}(\mathbf{b}, I(x))$.

Proof. This is a direct consequence of [V] prop. 6.5, where a similar statement is proved for an apartment $A \subset \mathcal{B}$, instead of a $SU(2, L)$-building $\mathbf{b} \subset \mathcal{B}$. □

8.6 Lemma. *Let $x \in Y_{\mathbf{b}}^{s}$ and let $A \subset \mathbf{b}$ be an apartment such that $\psi_{\mathbf{b}}(x) \in A$. Let $\rho_{\mathbf{b}}(x) := min\{v(\frac{x_0^2}{x_1 x_2}), 0\}$, where the x_i, $i = 0, 1, 2$ are the coordinates of $\mathbb{P}_L^2$ corresponding to A. Then $I_{\mathbf{b}}(x) := \{u \in \mathbf{b} \mid d_{\mathbf{b}}(\psi_{\mathbf{b}}(x), u) \leq -\rho_{\mathbf{b}}(x)\}$.*

Proof. For the convenience of the reader we give a proof here, even though this has been proved in [V] prop. 5.6.

Let $A \subset \mathbf{b}$ be an apartment containing $\psi_{\mathbf{b}}(x)$. Let x_0, x_1, x_2 be the coordinates of $\mathbb{P}_L^2$, such that the torus S belonging to A acts diagonally. Since $|\frac{g^* x_i}{x_i}| = 1$, $i = 1, 2$ and $g^* x_0 = x_0$ for all elements $g \in P_{\psi_{\mathbf{b}}(x)} \bigcap SU(2, L)$, the value of $\rho_{\mathbf{b}}(x)$ does not depend on the apartment $A \ni \psi_{\mathbf{b}}(x)$ used.

Let $t \in S$ be an element such that $|t^* x_1| = |t^* x_2|$ holds for the point $x \in Y_{\mathbf{b}}^{s}$. We replace the coordinates x_i by $t^* x_i$ for $i = 0, 1, 2$. Now the coordinates x_i are coordinates of $\mathbb{P}_{\mathbb{C}_p}^2$ instead of $\mathbb{P}_L^2$.

Let $s \in S$ be the diagonal element $diag(1, s_1, s_2)$ w.r.t. the coordinates x_i, $i = 0, 1, 2$ of $\mathbb{P}^2_{\mathbb{C}_p}$. The reduction $\bar{x}$ is semistable for the torus $S \otimes \ell$ at $s \cdot \psi_{\mathbf{b}}(x)$ if and only if $|s^* x_1^2| \leq max\{|x_0^2|, |x_1 x_2|\}$ and $|s^* x_2^2| \leq max\{|x_0^2|, |x_1 x_2|\}$. Furthermore, $|s^* x_1^2|, |s^* x_2^2| \leq max\{|x_0^2|, |x_1 x_2|\}$ if and only if $\rho_{\mathbf{b}}(x) \leq v(\frac{s_1}{s_2}) \leq -\rho_{\mathbf{b}}(x)$.

Since $d_{\mathbf{b}}(s \cdot \psi_{\mathbf{b}}(x), \psi_{\mathbf{b}}(x)) = v(\frac{s_1}{s_2})$, the reduction $\bar{x}$ of x is semistable for the torus $S \otimes \ell$ at $s \cdot \psi_{\mathbf{b}}(x) \in A \in \mathbf{b}$ if and only if $d_{\mathbf{b}}(s \cdot \psi_{\mathbf{b}}(x), \psi_{\mathbf{b}}(x)) \leq -\rho_{\mathbf{b}}(x)$.

Since $g \in P_{s \cdot \psi_{\mathbf{b}}(x)} \bigcap SU(2, L)$ acts only on the coordinates x_1 and x_2, the point x is also semistable in the reduction at $s \cdot \psi_{\mathbf{b}}(x)$ for the torus $gSg^{-1} \otimes \ell$. Hence x is semistable in the reduction at $s \cdot \psi_{\mathbf{b}}(x)$ for all apartments $A \ni s \cdot \psi_{\mathbf{b}}(x)$.

Therefore x is semistable in the reduction for the group $SU(2, L)$ at the point $u \in \mathbf{b}$ if and only if $d_{\mathbf{b}}(u, \psi_{\mathbf{b}}(x)) \leq -\rho_{\mathbf{b}}(x)$. □

8.7. A $SU(3, L)$-equivariant map $\psi_{\mathcal{B}} : Y^s \longrightarrow \mathcal{B}$. Let $x \in Y^s$ be a point. Since $Y^s \subset Y^s_{\mathbf{b}}$, the image $\psi_{\mathbf{b}}(x) \in \mathbf{b}$ is well-defined for any $SU(2, L)$-building $\mathbf{b} \subset \mathcal{B}$.

Let $A \subset \mathcal{B}$ be an apartment such that $r_{A,A}(x) = r(x)$. Then A is contained in a unique $SU(2, L)$-building $\mathbf{b} \subset \mathcal{B}$. Therefore $I_{\mathbf{b}}(x)$ and $\psi_{\mathbf{b}}(x)$ are well-defined. Let us consider the set $b_{\psi_{\mathbf{b}}(x)}$ containing the $SU(2, L)$-buildings $\mathbf{b}' \ni \psi_{\mathbf{b}}(x)$. Then $\{\rho_{\mathbf{b}'}(x) \mid \mathbf{b}' \in b_{\psi_{\mathbf{b}}(x)}\}$ obtains its infinum for some $SU(2, L)$-building $\mathbf{b}'' \in b_{\psi_{\mathbf{b}}(x)}$. Indeed, the compact group $P_{\psi_{\mathbf{b}}(x)} \subset SU(3, L)$ that stabilises the point $\psi_{\mathbf{b}}(x) \in \mathcal{B}$ acts transitively on the set $b_{\psi_{\mathbf{b}}(x)}$ and the function $\rho_{\mathbf{b}'}(x), \mathbf{b}' \in b_{\psi_{\mathbf{b}}(x)}$ is continuous. Without loss of generality we may assume that $\mathbf{b} = \mathbf{b}''$. Then $I(x) = I_{\mathbf{b}}(x) \subset \mathbf{b}$ and we define $\psi_{\mathcal{B}}(x) := \psi_{\mathbf{b}}(x)$. Therefore $\psi_{\mathcal{B}}(x)$ is the center of the interval of the interval of semistability of x (See [V] def. 6.4). A map $\psi^{\vee}_{\mathcal{B}} : Y^{s\vee} \to \mathcal{B}$ is defined analogously.

9 An admissable open subspace of $\Sigma_{\mathbf{b}}$

An open admissable subspace $\Sigma^{\circ}_{\mathbf{b}} \subset \Sigma_{\mathbf{b}}$ is defined. We define and describe in some detail an open admissable subspace $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} \subset \Sigma_{\mathbf{v},\mathbf{b}}$. The space $\Sigma^{\circ}_{\mathbf{b}}$ is obtained by glueing the spaces $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}}$ for the vertices $\mathbf{v} \in \mathbf{b}$. The space $\Sigma^{\circ}_{\mathbf{b}}$ will be used to construct the uniformising space in the next section.

9.1. Open admissable subspaces of $\Sigma_{\mathbf{b}}$ belonging to vertices v and edges e in b. For a $SU(2, L)$-building $\mathbf{b} \subset \mathcal{B}$, we consider the space $\Sigma_{\mathbf{b}}$

that belongs to the group $SU(2,L) \subset SU(3,L)$ that acts on $\mathbf{b}$. For each vertex $\mathbf{v} \in \mathbf{b}$ we define an admissable open subspace $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} \subset \Sigma_{\mathbf{v},\mathbf{b}}$.

If the vertex $\mathbf{v}$ is of type $\tau(\mathbf{v}) = 0$, then $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}}$ is defined as $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} := \Sigma_{\mathbf{v},\mathbf{b}} \bigcap \{x \in Y^s \mid d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v}) < 1\}$. If $\tau(\mathbf{v}) = 1$, then $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} := \Sigma_{\mathbf{v},\mathbf{b}} \bigcap \{z \in Y^{s\vee} \mid d_{\mathcal{B}}(\psi^{\vee}_{\mathcal{B}}(z), \mathbf{v}) < 1\}$.

Similarly, we define for edges $\mathbf{e} \ni \mathbf{b}$ two spaces. Let $\mathbf{v}_i \in \mathbf{e}$ be the vertex of type $\tau(\mathbf{v}_i) = i$ for $i = 0, 1$. Then $\Sigma^{\circ}_{\mathbf{e},\mathbf{b}} := \Sigma_{\mathbf{e},\mathbf{b}} \bigcap \{x \in Y^s \mid \psi_{\mathcal{B}}(x) \in \mathbf{e}, \psi_{\mathcal{B}}(x) \neq \mathbf{v}_0\}$ and $\Sigma^{\circ\vee}_{\mathbf{e},\mathbf{b}} := \Sigma^{\vee}_{\mathbf{e},\mathbf{b}} \bigcap \{z \in Y^{s\vee} \mid \psi^{\vee}_{\mathcal{B}}(z) \in \mathbf{e}, \psi^{\vee}_{\mathcal{B}}(z) \neq \mathbf{v}_1\}$.

9.2 Lemma. *Let $\mathbf{v} \in \mathbf{b}$ be a vertex of type $\tau(\mathbf{v}) = 0$ and let $x \in \Sigma^{\circ}_{\mathbf{v},\mathbf{b}}$. Then the following statements hold:*

i) If $\psi_{\mathcal{B}}(x) \neq \mathbf{v}$, then $\mathbf{v} \notin I(x)$.

ii) If $\psi_{\mathcal{B}}(x) = \mathbf{v}$, then $I(x) = \{\mathbf{v}\}$.

iii) Let $\mathbf{b}' \subset \mathcal{B}$ be a $SU(2,L)$-building that contains the vertex $\mathbf{v}$.

a) If $\psi_{\mathcal{B}}(x) \notin \mathbf{b}'$, then $I_{\mathbf{b}'}(x) = \{\mathbf{v}\}$.

b) If $\psi_{\mathcal{B}}(x) \in \mathbf{b}'$, then $I_{\mathbf{b}'}(x) = I(x)$.

Proof. To prove the first two statements of the lemma, we consider an element $g \in P_{\mathbf{v}}$ such that $\psi_{\mathcal{B}}(x) \in g(A)$ and $I(x) \subset g(\mathbf{b})$. Since $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v}) < 1$, such an element g exists.

Now $b_{\mathbf{v}}(x) = g^*x_0^{q+1} + g^*x_1 \cdot g^*x_2^q + g^*x_2 \cdot g^*x_1^q + \pi \cdot f(x) = 0$, where $f(x)$ is a homogeneous polynomial of degree $q+1$. Without loss of generality we may assume that $|\frac{g^*x_1}{g^*x_2}| \leq 1$. Then $(\frac{g^*x_0}{g^*x_2})^{q+1} + \frac{g^*x_1}{g^*x_2} + (\frac{g^*x_1}{g^*x_2})^q + \pi \cdot \frac{f(x)}{g^*x_2^{q+1}} = 0$. Since $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v}) < 1$ and $\psi_{g(\mathbf{b})}(x) = \psi_{\mathcal{B}}(x)$, we have $1 \geq |\frac{g^*x_1}{g^*x_2}| > |\pi|$. Therefore $|\frac{g^*x_0}{g^*x_2}|^{q+1} = |\frac{g^*x_1}{g^*x_2}|$ holds, unless $|\frac{g^*x_1}{g^*x_2}| = 1$ and $|\frac{g^*x_1}{g^*x_2} + (\frac{g^*x_1}{g^*x_2})^q| < 1$ holds.

In the latter case $\psi_{\mathcal{B}}(x) = \mathbf{v}$ and $\frac{g^*x_1}{g^*x_2} \equiv \omega \mod \pi$ for some $\omega \in L^{\circ}$ such that $\omega^q = -\omega$. We will show that this cannot occur. There exists an element $h \in SU(3,L)$ such that $h^*g^*x_1 = g^*x_1 - \omega g^*x_2$, $h^*g^*x_2 = g^*x_2$, $h^*g^*x_0 = g^*x_0$. Then $|\frac{h^*g^*x_1 h^*g^*x_2}{g^*x_1 g^*x_2}| < 1$. It follows that $\psi_{\mathcal{B}}(x) \notin gA$. In particular, $\psi_{\mathcal{B}}(x) \neq \mathbf{v}$. This contradicts our assumptions. Hence this cannot occur.

We conclude that $|\frac{g^*x_0}{g^*x_2}|^{q+1} = |\frac{g^*x_1}{g^*x_2}|$ holds. Multiplying both sides with $|\frac{g^*x_2}{g^*x_1}|^{(q+1)/2}$ gives $|\frac{g^*x_0^2}{g^*x_1 g^*x_2}|^{(q+1)/2} = |\frac{g^*x_2}{g^*x_1}|^{(q-1)/2}$. Otherwise stated $\rho_{g(\mathbf{b})}(x) = \frac{q-1}{q+1} \cdot v(\frac{g^*x_2}{g^*x_1}) = -\frac{q-1}{q+1} \cdot d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v})$.

Therefore if $\psi_{\mathcal{B}}(x) \neq \mathbf{v}$, then $\mathbf{v} \notin I_{g(\mathbf{b})}(x)$ and $I(x) = I_{g(\mathbf{b})}(x)$. If $\psi_{\mathcal{B}}(x) = \mathbf{v}$, then $I(x) = I_{g(\mathbf{b})}(x) = \{\mathbf{v}\}$. This proves statements (i) and (ii) of the lemma.

Statement (iii)a follows from statement (i) and the fact that $\mathbf{v} \in \mathbf{b}'$ is the vertex closest to $I(x)$ if $\psi_{\mathcal{B}}(x) \notin \mathbf{b}'$. Statement (iii)b is a direct consequence of statements (i) and (ii) of the lemma. □

9.3. Isotropic points. Let $\mathbf{v} \in \mathbf{b} \subset \mathcal{B}$ be a vertex of type $\tau(\mathbf{v}) = 0$ and let $[M_{\mathbf{v}}]$ be the corresponding equivalence class of L°-modules. Let $a \in \mathbb{P}(M_{\mathbf{v}})$ be an isotropic point such that the reduction of a is not contained in the reduction of the $\mathbb{P}^1_L$ given by $x_0 = 0$ belonging to $\mathbf{b}$. Then $a = (a_0, a_1, a_2)$ with $|\frac{a_0^2}{a_1 a_2}| = 1$.

The closed ball of radius r around a in $\Sigma_{\mathbf{v},\mathbf{b}}$ is defined by $B(a, r) := \{x \in \Sigma_{\mathbf{v},\mathbf{b}} \mid |\frac{h(x,a)}{a_0 \cdot x_0}| \leq r\}$. Since $x_0 \neq 0$ for $x \in \Sigma_{\mathbf{v},\mathbf{b}}$ this is well-defined. For a vertex $\mathbf{v} \in \mathbf{b} \subset \mathcal{B}$ we define the set $Iso(\mathbf{v}, \mathbf{b})$ as $Iso(\mathbf{v}, \mathbf{b}) := \{a \in \mathbb{P}^2(L) \mid h(a, a) = 0, |\frac{a_0^2}{a_1 a_2}| = 1\}$.

9.4 Proposition. *Let $\mathbf{b} \subset \mathcal{B}$ be a $SU(2, L)$-building and let $\mathbf{v} \in \mathbf{b}$ be a vertex and let $\mathbf{e} \in \mathbf{b}$ be an edge. Then the following statements hold:*

i) If $\tau(\mathbf{v}) = 0$, then $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} = \Sigma_{\mathbf{v},\mathbf{b}} - \bigcup_{a \in Iso(\mathbf{v},\mathbf{b})} B(a, |\pi|)$.

ii) If $\tau(\mathbf{v}) = 1$, then $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} = \Sigma_{\mathbf{v},\mathbf{b}}$.

iii) $\Sigma^{\circ}_{\mathbf{e},\mathbf{b}} = \Sigma_{\mathbf{e},\mathbf{b}}$ and $\Sigma^{\circ\vee}_{\mathbf{e},\mathbf{b}} = \Sigma^{\vee}_{\mathbf{e},\mathbf{b}}$.

Proof. Let us prove statement (i) of the proposition. We first show that the points $x \in \Sigma_{\mathbf{v},\mathbf{b}}$ such that $\psi_{\mathbf{b}}(x) \neq \mathbf{v}$ are contained in $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}}$. If $x \in \Sigma_{\mathbf{v},\mathbf{b}}$, then $\rho_{\mathbf{b}}(x) := min\{v(\frac{x_0^2}{x_1 x_2}), 0\} = v(\frac{x_0^2}{x_1 x_2}) = -\frac{q-1}{q+1} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi_{\mathbf{b}}(x))$.

Since by assumption $\psi_{\mathbf{b}}(x) \neq \mathbf{v}$, the value of $\rho_{\mathbf{b}}(x)$ is non-zero. Therefore $I_{\mathbf{b}}(x) = I(x) \bigcap \mathbf{b}$ holds. The convexity of $I(x)$ and the fact that $\mathbf{v} \notin I_{\mathbf{b}}(x)$ imply that $I(x) = I_{\mathbf{b}}(x)$ holds. Therefore $\psi_{\mathcal{B}}(x) = \psi_{\mathbf{b}}(x)$ and $d_{\mathcal{B}}(\mathbf{v}, \psi_{\mathcal{B}}(x)) < 1$. Hence $x \in \Sigma^{\circ}_{\mathbf{v},\mathbf{b}}$. One verifies that $x \notin B(a, |\pi|)$ for all $a \in Iso(\mathbf{v}, \mathbf{b})$.

Let us now consider the case where $\psi_{\mathbf{b}}(x) = \mathbf{v}$. Let $a \in Iso(\mathbf{v}, \mathbf{b})$ be the isotropic point $a = (a_0, a_1, a_2)$. Take $\alpha = \overline{(\frac{a_0}{a_2})}$ and $\beta = \overline{(\frac{a_2}{a_2})}$. Then one can define an element $g_a \in P_{\mathbf{v}} \subset SU(3, L)$ as follows: $g_a^* x_1 = x_1 + \alpha \cdot x_0 + \beta \cdot x_2$, $g_a^* x_2 = x_2$, $g_a^* x_0 = x_0 - \overline{\alpha} \cdot x_2$. Then $(g_a^* x_0, g_a^* x_1, g_a^* x_2) = (x_0 - \frac{a_0 x_2}{a_2}, \frac{h(x,a)}{\overline{a_2}}, x_2)$. In particular, for the point a, one has: $(g_a^* x_0, g_a^* x_1, g_a^* x_2) = (0, 0, a_2)$. The set of elements $\{g_a \in P_{\mathbf{v}} \mid a \in Iso(\mathbf{v},\mathbf{b})\}$ acts transitively on the halfapartments that start in the vertex $\mathbf{v}$ and that are not contained in $\mathbf{b}$.

Let us assume that $\psi_{\mathbf{b}}(x) = \mathbf{v}$ and that $x \notin Y^s$. Then $x \notin \Sigma^\circ_{\mathbf{v},\mathbf{b}}$ and we have to show that $x \in B(a, |\pi|)$ for some $a \in Iso(\mathbf{v},\mathbf{b})$. There exist a point $a \in \mathbb{P}^2(L)$ such that $h(a,a) = 0$ and $h(x,a) = 0$.

If $a \in Iso(\mathbf{v},\mathbf{b})$, then $g_a^* x_1 = \frac{h(x,a)}{\overline{a_2}} = 0$. Hence $x \in B(a, |\pi|)$. If $a \notin Iso(\mathbf{v},\mathbf{b})$, then $|a_0^2| < |a_1 a_2|$ and $|a_1 x_2 + a_2 x_1| < 1$. In particular, $\psi_{\mathbf{b}}(x) \neq \mathbf{v}$. This contradicts our assumptions. Hence this cannot occur and $x \in Y^s$ must hold.

Let us now assume that $\psi_{\mathbf{b}}(x) = \mathbf{v}$, $x \in Y^s$ and moreover, that $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v}) \geq 1$. Then $x \notin \Sigma^\circ_{\mathbf{v},\mathbf{b}}$ and we have to show that $x \in B(a, |\pi|)$ for some $a \in Iso(\mathbf{v},\mathbf{b})$. There exists an element $g_a \in SU(3, L)$ such that the apartment $g_a A$ contains $\psi_{\mathcal{B}}(x)$. Then $I_{g_a(\mathbf{b})}(x) = I(x) \bigcap g_a(\mathbf{b}) \subseteq I(x)$. We can choose the element g_a in such a way that the apartment $g_a A$ contains a point $r \in I(x)$, that has distance $R = max\{d_{\mathcal{B}}(u, \psi_{\mathcal{B}}(x)) \mid u \in I(x)\}$, the radius of $I(x)$ to the center $\psi_{\mathcal{B}}(x)$ of $I(x)$ and distance $d_{\mathcal{B}}(\mathbf{v}, \psi_{\mathcal{B}}(x)) + R$ to the vertex $\mathbf{v}$. Then $d_{\mathcal{B}}(\psi_{g_a(\mathbf{b})}(x), \mathbf{v}) \geq d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v})$ holds. In particular, $d_{\mathcal{B}}(\psi_{g_a(\mathbf{b})}(x), \mathbf{v}) \geq 1$. Therefore $v(\frac{g_a^* x_1}{g_a^* x_2}) = v(\frac{h(x,a)}{\overline{a_2} x_2}) \geq 1$. Hence $x \in B(a, |\pi|)$ and $x \notin \Sigma^\circ_{\mathbf{v},\mathbf{b}}$.

Next we show that if $\psi_{\mathbf{b}}(x) = \mathbf{v}$ and $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v})) < 1$, then for all $a \in Iso(\mathbf{v},\mathbf{b})$ the point $x \notin B(a, |\pi|)$. Indeed, since $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v})) < 1$, for an element g_a we have either $I(x) \bigcap g_a(\mathbf{b}) = \emptyset$ and $I_{g_a(\mathbf{b})}(x) = \mathbf{v}$ or $I_{g_a(\mathbf{b})}(x) = I(x)$ and $\psi_{g_a(\mathbf{b})}(x) = \psi_{\mathcal{B}}(x)$. In both cases $0 \leq v(\frac{g_a^* x_1}{g_a^* x_2}) = v(\frac{h(x,a)}{\overline{a_2} x_2}) < 1$ holds. In particular, $x \notin B(a, |\pi|)$. This concludes the proof of statement (i) of the proposition.

Let us now prove statement (ii). Let $\tau(\mathbf{v}) = 1$ and let $z \in \Sigma_{\mathbf{v},\mathbf{b}}$. Then $v(\frac{\pi z_0^{q+1}}{(\pi z_1 z_2)^{(q+1)/2}}) = v(\frac{a_{\mathbf{v},\mathbf{b}}((z_1,z_2))}{(\pi z_1 z_2)^{(q+1)/2}}) = -\frac{q-1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi^\vee_{\mathbf{b}}(z))$. Therefore $v(\frac{z_0^2}{\pi z_1 z_2}) = -\frac{2}{q+1} - \frac{q-1}{q+1} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi^\vee_{\mathbf{b}}(z))$. So $v(\frac{z_0^2}{z_1 z_2}) = \frac{q-1}{q+1} - \frac{q-1}{q+1} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi^\vee_{\mathbf{b}}(z)) > 0$ holds, since $d_{\mathbf{b}}(\mathbf{v}, \psi^\vee_{\mathbf{b}}(z)) < 1$. From this one concludes that $I^\vee_{\mathbf{b}}(z) = \{\psi^\vee_{\mathbf{b}}(z)\}$ holds. Therefore $\psi^\vee_{\mathcal{B}}(z) = \psi^\vee_{\mathbf{b}}(z)$ holds and $z \in \Sigma^\circ_{\mathbf{v},\mathbf{b}}$. Hence $\Sigma^\circ_{\mathbf{v},\mathbf{b}} = \Sigma_{\mathbf{v},\mathbf{b}}$ if $\tau(\mathbf{v}) = 1$.

Let us now prove statement (iii). If $x \in \Sigma_{\mathbf{e},\mathbf{b}}$ then $\psi_{\mathbf{b}}(x) \in \mathbf{e}$ and $\psi_{\mathbf{b}}(x) \neq \mathbf{v}$. Then $I(x) = I_{\mathbf{b}}(x)$ and $\psi_{\mathcal{B}}(x) = \psi_{\mathbf{b}}(x) \in \mathbf{e}$. Hence $x \in \Sigma^\circ_{\mathbf{e},\mathbf{b}}$. Therefore $\Sigma^\circ_{\mathbf{e},\mathbf{b}} = \Sigma_{\mathbf{e},\mathbf{b}}$ holds. For $z \in \Sigma^\vee_{\mathbf{e},\mathbf{b}}$ one again verifies that $\psi^\vee_{\mathcal{B}}(z) = \psi^\vee_{\mathbf{b}}(z)$ holds. Therefore $\Sigma^{\circ\vee}_{\mathbf{e},\mathbf{b}} = \Sigma^\vee_{\mathbf{e},\mathbf{b}}$ holds.

This concludes the proof of the proposition. □

9.5 Proposition. *Let* $\Sigma^\circ_{\mathbf{b}} := \bigcup_{\mathbf{v}\in\mathbf{b}} \Sigma^\circ_{\mathbf{v},\mathbf{b}} / \sim$*, where* $\sim$ *denotes the equivalence* *relation obtained by applying the identifications* $\Sigma^{\circ}_{\mathbf{e},\mathbf{b}} \cong \Sigma^{\circ,\vee}_{\mathbf{e},\mathbf{b}}$ *for all edges* $\mathbf{e} \in \mathbf{b}$. *Then the following four statements hold:*

i) The space $\Sigma^{\circ}_{\mathbf{b}}$ *is a well-defined rigid analytical variety.*

ii) The space $\Sigma^{\circ}_{\mathbf{b}}$ *is an open admissable subspace of* $\Sigma_{\mathbf{b}}$.

iii) $\Sigma^{\circ}_{\mathbf{b}} = \Sigma_{\mathbf{b}} - \bigcup_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=0} \bigcup_{c\in Iso(\mathbf{v},\mathbf{b})} B(c,|\pi|)$.

Proof. Statement (i) is clear from the construction. The second statement follows from the fact that the spaces $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} \subset \Sigma_{\mathbf{v},\mathbf{b}}$ are open admissable subspaces. The third statement follows from the description of the spaces $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}}$ given in the proposition above. □

9.6 Proposition. *Let* $\mathbf{v} \in \mathbf{b}$ *be a vertex of type* $\tau(\mathbf{v})$. *Then:*

i) If $\tau(\mathbf{v}) = 0$, *then* $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} = \Sigma_{\mathbf{v},\mathbf{b}} \bigcap \{x \in Y^s \mid d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{b}) < 1\}$.

ii) If $\tau(\mathbf{v}) = 1$, *then* $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} = \Sigma_{\mathbf{v},\mathbf{b}} \bigcap \{z \in Y^{s\vee} \mid d_{\mathcal{B}}(\psi^{\vee}_{\mathcal{B}}(z), \mathbf{b}) < 1\})$.

Proof. Let us prove the first statement of the proposition. For a point $x \in \Sigma^{\circ}_{\mathbf{v},\mathbf{b}}$, the inequality $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{b}) \le d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v}) < 1$ holds. Hence the inclusion $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} \subset \{x \in Y^s \mid d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{b}) < 1\}$ holds.

Now let $x \in \Sigma_{\mathbf{v},\mathbf{b}} - \Sigma^{\circ}_{\mathbf{v},\mathbf{b}}$. Then $x \in B(a,|\pi|)$ for some isotropic point $a \in Iso(\mathbf{v},\mathbf{b})$. The isotropic point a corresponds to an edge $\mathbf{e} \ni \mathbf{v}$ that is not contained in $\mathbf{b}$. In particular, if $x \in Y^s$, then the path from $\psi_{\mathcal{B}}(x)$ to the vertex $\mathbf{v}$ contains the edge $\mathbf{e}$. Therefore the vertex $\mathbf{v} \in \mathbf{b}$ is such that the equality $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{b}) = d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v})$ holds. It follows that $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} = \Sigma_{\mathbf{v},\mathbf{b}} \bigcap \{x \in Y^s \mid d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{b}) < 1\}$ holds.

The second statement is straightforward, since $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} = \Sigma_{\mathbf{v},\mathbf{b}}$ if $\tau(\mathbf{v}) = 1$. This concludes the proof of the proposition. □

10 The uniformising space

Let $\Gamma \subset SU(3,L)$ be a discrete co-compact subgroup that preserves an almost complete transversal system $\mathcal{T}$ of $SU(2,L)$-buildings. A one dimensional rigid analytical space $\mathcal{Y}$ (which depends on Γ and $\mathcal{T}$) is constructed on which Γ acts discretely and the quotient $\mathcal{Y}/\Gamma$ is a proper algebraic curve.

The spaces $\Sigma^{\circ}_{\mathbf{b}}$ for $\mathbf{b} \in \mathcal{T}$ are glued together into a space $\mathcal{Y}^{\circ}$ on which the group Γ acts discretely. The quotient $\mathcal{Y}^{\circ}/\Gamma$ is not proper. To compactify the quotient, one adds to the space $\mathcal{Y}^\circ$ a suitable admissable space $\mathcal{Y}_{\mathbf{v}}$ for each vertex $\mathbf{v} \in \mathcal{B}$ that is not contained in a $SU(2,L)$-building $\mathbf{b} \in \mathcal{T}$. The spaces $\mathcal{Y}_{\mathbf{v}}$ that are added are isomorphic to open admissable subsets of Ω_1.

The result is the space $\mathcal{Y}$ on which Γ acts discretely with proper quotient $\mathcal{Y}/\Gamma$. A Γ-invariant pure affinoid covering of $\mathcal{Y}$ and a description of the reduction of $\mathcal{Y}$ w.r.t. this covering is given.

10.1. Polynomials for the vertices contained in $|\mathcal{T}|$. Let $\Gamma \subset SU(3,L)$ be a discrete co-compact subgroup that admits a Γ-invariant almost complete transversal system of $SU(2,L)$-buildings $\mathcal{T}$. We give extra conditions on the homogeneous polynomials $a_{\mathbf{v},\mathbf{b}}(z)$ and $b_{\mathbf{v}}(x)$ associated to the vertices $\mathbf{v} \in |\mathcal{T}| := \bigcup_{\mathbf{b}\in\mathcal{T}} \mathbf{b}$. These extra conditions allow us to glue the open admissable subsets $\Sigma_{\mathbf{b}}^\circ \subset \Sigma_{\mathbf{b}}$ for $\mathbf{b} \in \mathcal{T}$ together.

Let $\mathbf{b}_0 \in \mathcal{T}$ be an $SU(2,L)$-building. Let $\mathbf{v}_0 \in \mathbf{b}_0$ be the vertex corresponding to the equivalence class $[M_{\mathbf{v}_0}]$, $M_{\mathbf{v}_0} =< e_0, e_1, e_2 >$ and let $\mathbf{v}_1 \in \mathbf{b}_0$ be the vertex corresponding to the equivalence class $[M_{\mathbf{v}_1}]$, $M_{\mathbf{v}_1} =< e_0, e_1, \pi^{-1}e_2 >$. The homogeneous polynomial $b_{\mathbf{v}_0}(x)$ is such that $b_{\mathbf{v}_0}(x) \equiv x_0^{q+1} + x_1 x_2^q + x_2 x_1^q \mod \pi$. The homogeneous polynomial $a_{\mathbf{v}_1,\mathbf{b}_0}(z)$ is of degree $q+1$ in the coordinates z_1 and z_2 such that $a_{\mathbf{v}_1,\mathbf{b}_0}(z) \equiv z_1(\pi z_2)^q + z_1^q(\pi z_2) \mod \pi$ holds. Let $\mathbf{v} = g(\mathbf{v}_0)$. Then $b_{\mathbf{v}}(x) \equiv g^* b_{\mathbf{v}_0}(x) \mod \pi$. Let $\mathbf{v}' \in \mathbf{b} \in \mathcal{T}$ and let $g \in SU(3,L)$ be such that $\mathbf{v}' = g(\mathbf{v}_1)$ and $\mathbf{b} = g(\mathbf{b}_0)$. Then the polynomial $a_{\mathbf{v}',\mathbf{b}}(z)$ is homogeneous of degree $q+1$ in the coordinates $g^* z_1$ and $g^* z_2$ such that $a_{\mathbf{v}',\mathbf{b}}(z) \equiv g^* a_{\mathbf{v}_1,\mathbf{b}_0}(z) \mod \pi$ holds.

In this way we associate to each vertex $\mathbf{v} \in |\mathcal{T}| \subset \mathcal{B}$ of type $\tau(\mathbf{v}) = 0$ ($\tau(\mathbf{v}) = 1$) that is contained in a $SU(2,L)$-building $\mathbf{b} \in \mathcal{T}$ a homogeneous polynomial $b_{\mathbf{v}}(x)$ ($a_{\mathbf{v},\mathbf{b}}(z)$) of degree $q+1$ such that the following two conditions hold:

i) If $\tau(\mathbf{v}) = 0$, then $b_{\gamma(\mathbf{v})}(x) = \gamma^* b_{\mathbf{v}}(x)$ for all $\gamma \in \Gamma$.

ii) If $\tau(\mathbf{v}) = 1$, then $a_{\gamma(\mathbf{v}),\gamma(\mathbf{b})}(z) = \gamma^* a_{\mathbf{v},\mathbf{b}}(z)$ for all $\gamma \in \Gamma$.

Let $H_{\mathbf{b}} \subset SU(3,L)$ be the stabiliser of the $SU(2,L)$-building $\mathbf{b} \in \mathcal{T}$. Let the group $\Gamma \subset SU(3,L)$ be arithmetic and defined over a number field $\mathcal{K} \supset \mathbb{Q}$. Then $\Gamma = G_\Lambda(\mathcal{O}_{\mathcal{L}}[1/s])$ as a group defined over $\mathcal{K}$ preserving a unitary form h_0 and a $\mathcal{O}_{\mathcal{L}}[1/s]$-lattice $\Lambda[1/s]$. Even if one assumes that the hermitian lattice Λ is unimodular, the group $\Gamma \bigcap H_{\mathbf{b}}$ is in general not defined by a lattice that is unimodular. Indeed, the group $\Gamma \bigcap H_{\mathbf{b}}$ is the stabiliser in Γ of a vector $x_{\mathbf{b}} \in \Lambda[1/s]$. In general $h_0(x_{\mathbf{b}}, x_{\mathbf{b}}) \neq 1$. Then the lattice

$< x_{\mathbf{b}} > \oplus < x_{\mathbf{b}}^{\perp} \bigcap \Lambda[1/s] >$ defining the group $\Gamma \bigcap H_{\mathbf{b}}$ is not unimodular. Therefore the polynomials $a_{\mathbf{v},\mathbf{b}}(z)$ cannot be defined over the number field $\mathcal{L} \supset \mathcal{K}$. They are defined over the field L.

10.2 Lemma. *Let $\mathbf{v} \in \mathcal{B}$ be a vertex of type $\tau(\mathbf{v}) = 0$ and let $\Gamma_{\mathbf{v}} \subset \Gamma$ be the stabiliser of the vertex $\mathbf{v}$. Then there exists a polynomial $b_{\mathbf{v}}(x)$ as above such that $\gamma^* b_{\mathbf{v}}(x) = b_{\mathbf{v}}(x)$ for all $\gamma \in \Gamma_{\mathbf{v}}$.*

Proof. Since the arithmetic group $\Gamma \subset SU(3, L)$ is co-compact, the elements of the group $\Gamma_{\mathbf{v}}$ are semisimple. Therefore the order of $\Gamma_{\mathbf{v}}$ is not divisible by the characteristic of the residue field of K. One can always replace a homogeneous polynomial $b_{\mathbf{v}}(x)$ by the homogeneous polynomial $\frac{1}{|\Gamma_{\mathbf{v}}|} \sum_{\gamma \in \Gamma_{\mathbf{v}}} \gamma^* b_{\mathbf{v}}(x)$ to obtain a polynomial that satisfies the lemma. □

10.3. Construction of the rigid analytic space $\mathcal{Y}^{\circ}$ for $|\mathcal{T}| := \bigcup_{\mathbf{b} \in \mathcal{T}} \mathbf{b}$. The space $\mathcal{Y}^{\circ}$ is defined by associating to each vertex $\mathbf{v} \in |\mathcal{T}|$ a analytic space $\mathcal{Y}_{\mathbf{v}}$ and glueing them together along open admissable subspaces $\mathcal{Y}_{\mathbf{e}}$ and $\mathcal{Y}_{\mathbf{e}}^{\vee}$ corresponding to edges $\mathbf{e} \in |\mathcal{T}|$.

The spaces $\mathcal{Y}_{\mathbf{v}}$ for $\mathbf{v} \in |\mathcal{T}|$ are defined as $\mathcal{Y}_{\mathbf{v}} := \Sigma^{\circ}_{\mathbf{v},\mathbf{b}}$. Here $\mathbf{b} \in \mathcal{T}$ is a $SU(2, L)$-building that contains the vertex $\mathbf{v}$. If $\tau(\mathbf{v}) = 1$, then the building $\mathbf{b} \ni \mathbf{v}$ is unique. If $\tau(\mathbf{v}) = 0$, then our condition i) on the polynomial $b_{\mathbf{v}}(x)$ ensures us that the space $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}}$ does not depend on the choice of $\mathbf{b} \ni \mathbf{v}$. Clearly, $\mathcal{Y}_{\mathbf{v}} \subset Y^s$ if $\tau(\mathbf{v}) = 0$ and $\mathcal{Y}_{\mathbf{v}} \subset Y^{s\vee}$ if $\tau(\mathbf{v}) = 1$.

For the edges $\mathbf{e} \in |\mathcal{T}|$ we define the spaces $\mathcal{Y}_{\mathbf{e}} := \Sigma^{\circ}_{\mathbf{e},\mathbf{b}}$ and $\mathcal{Y}^{\vee}_{\mathbf{e}} := \Sigma^{\circ\vee}_{\mathbf{e},\mathbf{b}}$. Let $\mathbf{v}, \mathbf{v}' \in \mathbf{e}$ be the vertices of type $\tau(\mathbf{v}) = 0$ and type $\tau(\mathbf{v}') = 1$. Then $\mathcal{Y}_{\mathbf{e}} \subset \mathcal{Y}_{\mathbf{v}}$ and $\mathcal{Y}^{\vee}_{\mathbf{e}} \subset \mathcal{Y}_{\mathbf{v}'}$ are open admissable subspaces.

Since each edge $\mathbf{e} \in |\mathcal{T}|$ is contained in a unique $SU(2, L)$-building $\mathbf{b} \in \mathcal{T}$ and $\mathcal{Y}_{\mathbf{e}} \cong \Sigma_{\mathbf{e},\mathbf{b}}$ and $\mathcal{Y}^{\vee}_{\mathbf{e}} \cong \Sigma^{\vee}_{\mathbf{e},\mathbf{b}}$, we can identify $\mathcal{Y}_{\mathbf{e}} \cong \mathcal{Y}^{\vee}_{\mathbf{e}}$ by using the same identifications used to identify $\Sigma_{\mathbf{e},\mathbf{b}} \cong \Sigma^{\vee}_{\mathbf{e},\mathbf{b}}$.

10.4 Proposition. *Let $\mathcal{Y}^{\circ} := \bigcup_{\mathbf{v} \in |\mathcal{T}|} \mathcal{Y}_{\mathbf{v}} / \sim$, where $\sim$ is the equivalence relation that identifies $\mathcal{Y}_{\mathbf{e}} \cong \mathcal{Y}^{\vee}_{\mathbf{e}}$ for all edges $\mathbf{e} \in |\mathcal{T}|$. Then:*

i) The space $\mathcal{Y}^{\circ}$ is a well-defined rigid analytical space.

ii) $\mathcal{Y}^{\circ} = \bigcup_{\mathbf{b} \in \mathcal{T}} \Sigma^{\circ}_{\mathbf{b}}$.

Proof. The identifications $\mathcal{Y}_{\mathbf{e}} = \Sigma_{\mathbf{e},\mathbf{b}} \cong \Sigma^{\vee}_{\mathbf{e},\mathbf{b}} = \mathcal{Y}^{\vee}_{\mathbf{e}}$ are well-defined for $\mathbf{e} \in \mathbf{b}$. Since each edge $\mathbf{e} \in \bigcup_{\mathbf{b} \in \mathcal{T}} \mathbf{b}$ is contained in a single building $\mathbf{b} \in \mathcal{T}$ and the spaces $\mathcal{Y}_{\mathbf{e}}$ are all disjoint, as are all the spaces $\mathcal{Y}^{\vee}_{\mathbf{e}}$, we can apply the identifications simultaneously for all edges $\mathbf{e} \in |\mathcal{T}|$. The result is a well-defined rigid analytic space $\mathcal{Y}^\circ$.

The second statement of the proposition follows from the fact that the identifications used to define the space $\mathcal{Y}^\circ$ when restricted to the spaces $\mathcal{Y}_\mathbf{v}$ with $\mathbf{v} \in \mathbf{b}$ for a $SU(2, L)$-building $\mathbf{b} \in \mathcal{T}$ define the space $\Sigma^\circ_\mathbf{b}$. □

10.5 Proposition. *Let $\mathbf{v} \in |\mathcal{T}|$ be a vertex of type $\tau(\mathbf{v}) = 0$. Let $\mathbf{b}_i \in \mathcal{T}$, $i = 1, \ldots, s$ be the $SU(2, L)$-buildings that contain the vertex $\mathbf{v}$. Let $H_i \cong S(U(1, L) \times U(2, L)) \subset SU(3, L)$ be the stabiliser of $\mathbf{b}_i$ and let $a_i \in \mathbb{P}(L)$ be the point that is fixed by the group H_i for $i = 1, \ldots, s$. Then there exist open affine subvarieties $X_i \subset \mathbb{P}$, $i = 1, \ldots, s$ such that the following statements hold:*

i) $a_i \in X_i$ *for* $i = 1, \ldots, s$.

ii) The intersection of the reduction $X_i \otimes \ell$ and the hermitian curve given by $b_\mathbf{v}(x) \equiv 0 \mod \pi$ in $\mathbb{P}(M_\mathbf{v}) \otimes \ell$ is an open affine subvariety of the hermitian curve.

iii) Let $R_\mathbf{v} : \mathbb{P}(M_\mathbf{v}) \to \mathbb{P}(M_\mathbf{v} \otimes \ell)$ be the reduction map. Let $\mathbf{e} \in \mathbf{b}_i$ be an edge containing the vertex $\mathbf{v}$. We consider the space $\mathcal{Y}_\mathbf{e}$ as a analytic subspace of $\mathbb{P}(M_\mathbf{v}) = \mathbb{P}$ and the affine varieties $X_j \otimes \ell$ as subspaces of $\mathbb{P}(M_\mathbf{v} \otimes \ell)$ for $j = 1, \ldots, s$. Then the following statements hold:

a) $\mathcal{Y}_\mathbf{e} \bigcap R_\mathbf{v}^{-1}(X_i \otimes \ell) = \emptyset$.

b) $\mathcal{Y}_\mathbf{e} \subset R_\mathbf{v}^{-1}(X_j \otimes \ell)$ *for* $j \neq i$, $j = 1, \ldots, s$.

iv) Let $X_{\mathbf{v},\mathbf{b}_i} \subset \mathbb{P}(M_\mathbf{v}) = \mathbb{P}$ be the open admissable analytical subspace $X_{\mathbf{v},\mathbf{b}_i} := R_\mathbf{v}^{-1}(\bigcap_{j\neq i} X_j \otimes \ell)$ if $s > 1$. If $s = 1$, then we define $X_{\mathbf{v},\mathbf{b}_1} := \mathbb{P}(M_\mathbf{v})$. Let $\mathbf{e} \in |\mathcal{T}|$ be an edge such that $\mathbf{v} \in \mathbf{e}$. Then the analytical space $\mathcal{Y}_\mathbf{e}$ is contained in the open admissable subspace $X_{\mathbf{v},\mathbf{b}_i} \subset \mathbb{P}$ if and only if $\mathbf{e} \in \mathbf{b}_i$.

Proof. For the point $a_i = (x_0, 0, 0)$, we take as X_i the affine variety defined by $x_0 \neq 0$. Let $g_{i,j} \in P_\mathbf{v} \subset SU(3, L)$ be an element such that $g_{i,j}(a_i) = a_j$. Then $g_{i,j}(\mathbf{b}_i) = \mathbf{b}_j$ holds for $j \neq i$. We define X_j by $X_j := g_{i,j}(X_i)$ for $j \neq i$. We will show that the thus obtained varieties X_j satisfy the proposition.

By construction statement (i) of the proposition holds. On the hermitian curve given by $b_\mathbf{v}(x) \equiv 0 \mod \pi$ in $\mathbb{P}(M_\mathbf{v}) \otimes \ell$, the intersection is again given by $\overline{x_0} \neq 0$ and is affine. Therefore statement (ii) holds.

Let us now prove statement (iii) of the proposition. The reduction $R_{\mathbf{v}}(\mathcal{Y}_{\mathbf{e}}) \subset \mathbb{P}(M_{\mathbf{v}} \otimes \ell)$ consists of a single ℓ-valued isotropic point. This is the isotropic point that corresponds to the edge $\mathbf{e} \in \mathcal{B}$. We may assume that $R_{\mathbf{v}}(\mathcal{Y}_{\mathbf{e}}) = \{(0, x_1, 0)\}$ holds. Since $\overline{x_0} = 0$ holds on $R_{\mathbf{v}}(\mathcal{Y}_{\mathbf{e}})$, it follows that the intersection $R_{\mathbf{v}}(\mathcal{Y}_{\mathbf{e}}) \bigcap X_i \otimes \ell$ is empty. Hence statement iii)a of the proposition holds.

To prove statement iii)b, we take again the element $g_{i,j} \in P_{\mathbf{v}} \subset SU(3, L)$ such that $g_{i,j}(\mathbf{b}_i) = \mathbf{b}_j$ for $j \neq i$. Then $\overline{g_{i,j}^* x_0} \neq 0$ holds on $R_{\mathbf{v}}(\mathcal{Y}_{\mathbf{e}})$. Therefore $R_{\mathbf{v}}(\mathcal{Y}_{\mathbf{e}}) \subset X_j \otimes \ell$. From this statement iii)b of the proposition follows.

Statement (iv) of the proposition is a direct consequence of statement (iii). □

10.6. Identification of coordinates in $\mathbb{P}$ and $\mathbb{P}^{\vee}$. The proposition above shows that around a vertex $\mathbf{v} \in |\mathcal{T}|$ of type $\tau(\mathbf{v}) = 0$, one can simplify the construction of the analytical space. On each of the open analytic subvarieties $X_{\mathbf{v},\mathbf{b}_i} \subset \mathbb{P}$ one can identify the coordinates of the line fixed by the group H_i in $\mathbb{P}$ with the coordinates of the line in $\mathbb{P}^{\vee}$ fixed by the group H_i. The identification of coordinates is given by a suitable translate of the equation $x_1 z_2 + x_2 z_1 = 0$. As a consequence one blows up the points in $\mathbb{P}$ and $\mathbb{P}^{\vee}$ that are fixed by the stabilisers H_i of the $SU(2, L)$-buildings $\mathbf{b}_i \in \mathcal{T}$ that contain the vertex $\mathbf{v}$ and identifies the exceptional lines in $\mathbb{P}$ (resp. $\mathbb{P}^{\vee}$) with the corresponding ordinary lines in $\mathbb{P}^{\vee}$ (resp. $\mathbb{P}$).

Let us now briefly discuss the situation for two $SU(2, L)$-buildings $\mathbf{b}_1, \mathbf{b}_2 \ni \mathbf{v}$ that do not intersect transversally at $\mathbf{v}$. Then they have an edge $\mathbf{e} \ni \mathbf{v}$ in common. Let the affine subvarieties $X_1 \subset \mathbb{P}$ and $X_2 \subset \mathbb{P}$ be defined as above. The isotropic point $R_{\mathbf{v}}(\mathcal{Y}_{\mathbf{e}})$ that corresponds to the edge $\mathbf{e}$ is neither contained in $X_1 \otimes \ell$ nor in $X_2 \otimes \ell$. Therefore $\mathcal{Y}_{\mathbf{e}}$ is not contained in $R_{\mathbf{v}}^{-1}(X_1 \otimes \ell)$ nor in $R_{\mathbf{v}}^{-1}(X_2 \otimes \ell)$. One needs to use another affine subvariety $Z_{\mathbf{e}} \subset \mathbb{P}(M_{\mathbf{v}}) = \mathbb{P}$ such that $\mathcal{Y}_{\mathbf{e}} \subset R_{\mathbf{v}}^{-1}(Z_{\mathbf{e}} \otimes \ell)$. On the open analytic space $R_{\mathbf{v}}^{-1}(Z_{\mathbf{e}} \otimes \ell)$ that contains the space $\mathcal{Y}_{\mathbf{e}}$ one must make a choice of which coordinates to use, i.e. those belonging to the building $\mathbf{b}_1$ or to the building $\mathbf{b}_2$. Therefore one has two choices for the coordinates that can be used to define the analytic spaces $\mathcal{Y}_{\mathbf{e}}^{\vee}$ and $\mathcal{Y}_{\mathbf{v}'}$, where $\mathbf{v}' \in \mathbf{e}$ is the vertex different from $\mathbf{v}$. One somehow has to make a Γ-invariant choice of which of the buildings $\mathbf{b}_1$ or $\mathbf{b}_2$ one wants to use to define these spaces. This might not be impossible, but it does make the construction somewhat more complicated. We will not follow this road.

10.7 Proposition. *Let $\mathbf{b}, \mathbf{b}' \in \mathcal{T}$ be two distinct $SU(2, L)$-buildings and let*

$\Sigma^\circ_{\mathbf{b}}, \Sigma^\circ_{\mathbf{b}'} \subset \mathcal{Y}^\circ$ *be the associated subspaces. Then exactly one the following two statements hold:*

a) The intersection $\mathbf{b} \bigcap \mathbf{b}'$ *is a vertex* $\mathbf{v}$ *and* $\Sigma^\circ_{\mathbf{b}} \bigcap \Sigma^\circ_{\mathbf{b}'} = \mathcal{Y}_{\mathbf{v}} = \Sigma^\circ_{\mathbf{v},\mathbf{b}} = \Sigma^\circ_{\mathbf{v},\mathbf{b}'}$.

b) The intersection $\mathbf{b} \cap \mathbf{b}'$ *is empty and the intersection* $\Sigma^\circ_{\mathbf{b}} \bigcap \Sigma^\circ_{\mathbf{b}'} = \emptyset$.

Proof. Let $\mathcal{Y}_{\mathbf{v}'} \subset \mathcal{Y}^\circ$. One verifies that $\mathcal{Y}_{\mathbf{v}'} \bigcap \Sigma^\circ_{\mathbf{b}} \neq \emptyset$ if and only if $\mathbf{v}' \in \mathbf{b}$. From this and the definition of a transversal sytem the proposition follows. □

10.8. A map $\psi^{\mathcal{Y}^\circ}_{\mathcal{B}} : \mathcal{Y}^\circ \to \mathcal{B}$. We define a map $\psi^{\mathcal{Y}^\circ}_{\mathcal{B}} : \mathcal{Y}^\circ \to \mathcal{B}$ by taking:

$$\psi^{\mathcal{Y}^\circ}_{\mathcal{B}} = \begin{cases} \psi_{\mathcal{B}} & \text{for points contained in } \bigcup_{\mathbf{v}\in|\mathcal{T}|,\tau(\mathbf{v})=0} \mathcal{Y}_{\mathbf{v}} \\ \psi^\vee_{\mathcal{B}} & \text{for points contained in } \bigcup_{\mathbf{v}\in|\mathcal{T}|,\tau(\mathbf{v})=1} \mathcal{Y}_{\mathbf{v}}. \end{cases}$$

To avoid making the notation unnecessary complex, we do not distinguish between the use of the coordinates x_i and z_i, $i = 0, 1, 2$ for this map. In the propostion below we show that this is well-defined.

10.9 Proposition. *The following statements hold:*

i) The map $\psi^{\mathcal{Y}^\circ}_{\mathcal{B}}$ *is well-defined and* Γ*-equivariant.*

ii) $\Sigma^\circ_{\mathbf{b}} = \{x \in \mathcal{Y}^\circ \mid d_{\mathcal{B}}(\psi^{\mathcal{Y}^\circ}_{\mathcal{B}}(x), \mathbf{b}) < 1\} \subset \mathcal{Y}^\circ$

iii) If $x \in \mathcal{Y}^\circ$*, then* $d_{\mathcal{B}}(\psi^{\mathcal{Y}^\circ}_{\mathcal{B}}(x), |\mathcal{T}|) < 1$.

iv) $\psi^{\mathcal{Y}^\circ}_{\mathcal{B}}(\mathcal{Y}^\circ) = \{u \in \mathcal{B}(\mathbb{Q}) \mid d_{\mathcal{B}}(u, |\mathcal{T}|) < 1\} = \mathcal{B}(\mathbb{Q}) - \{\mathbf{v} \mid \mathbf{v} \notin \bigcup_{\mathbf{b}\in\mathcal{T}} \mathbf{b}\}$.

Proof. To show that the map $\psi^{\mathcal{Y}^\circ}_{\mathcal{B}}$ is well-defined we have to show that at edges $\mathbf{e} \in |\mathcal{T}|$ the equality $\psi_{\mathcal{B}}(x) = \psi^\vee_{\mathcal{B}}(z)$ holds, whenever $x \in \mathcal{Y}_{\mathbf{e}}$ is identified with $z \in \mathcal{Y}^\vee_{\mathbf{e}}$ in $\mathcal{Y}^\circ$. We observe that if $\mathbf{b} \in \mathcal{T}$ contains the edge $\mathbf{e}$, then $\psi_{\mathcal{B}}(x) = \psi_{\mathbf{b}}(x)$ and $\psi^\vee_{\mathcal{B}}(z) = \psi^\vee_{\mathbf{b}}(z)$. Since $g^*x_1 g^*z_2 + g^*x_2 g^*z_1 = 0$ holds and $\psi_{\mathbf{b}}(x) = v(\frac{g^*x_1}{g^*x_2})$ and $\psi^\vee_{\mathbf{b}}(z) = v(\frac{g^*z_1}{g^*z_2})$ after a suitable identification of the apartment $A \subset \mathbf{b}$, $A \ni \mathbf{e}$ with the real line $\mathbb{R}$, we conclude that $\psi^{\mathcal{Y}^\circ}_{\mathcal{B}}(x) = \psi_{\mathbf{b}}(x) = \psi^\vee_{\mathbf{b}}(z) = \psi^{\mathcal{Y}^\circ}_{\mathcal{B}}(z)$ holds.

The map $\psi^{\mathcal{Y}^\circ}_{\mathcal{B}}$ is Γ-equivariant by construction.

Statements (ii) and (iii) of the proposition follow directly from the fact that $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v}) < 1$ for $x \in \mathcal{Y}_{\mathbf{v}}$. The final statement follows from the fact that $\mathcal{Y}_{\mathbf{e}} = \Sigma^{\circ}_{\mathbf{e},\mathbf{b}} = \Sigma_{\mathbf{e},\mathbf{b}}$. Therefore $\psi^{\mathcal{Y}^{\circ}}_{\mathcal{B}}(\mathcal{Y}_{\mathbf{e}}) = \psi_{\mathbf{b}}(\Sigma_{\mathbf{e},\mathbf{b}})$ and $\psi_{\mathbf{b}}(\Sigma_{\mathbf{e},\mathbf{b}}) = \mathbf{e} \bigcap \mathbb{Q} - \{\mathbf{v}_1, \mathbf{v}_2\}$. Here $\mathbf{v}_1$ and $\mathbf{v}_2$ are the vertices in $\mathbf{e}$. Clearly, $\mathbf{v} \in \psi^{\mathcal{Y}^{\circ}}_{\mathcal{B}}(\mathcal{Y}_{\mathbf{v}})$. Therefore the image of $\psi^{\mathcal{Y}^{\circ}}_{\mathcal{B}}$ contains all rational points of the building $\mathcal{B}$, except for the vertices $\mathbf{v} \in \mathcal{B} - |\mathcal{T}|$. □

10.10. Analytic spaces for the vertices v and edges e not contained in $|\mathcal{T}|$. For each vertex $\mathbf{v} \in \mathcal{B}$ that is not contained in $|\mathcal{T}| = \bigcup_{\mathbf{b} \in \mathcal{T}} \mathbf{b}$, we define an analytic subspace $\mathcal{Y}_{\mathbf{v}} \subset Y^{s\vee}$. Let $g_{\mathbf{v}} \in SU(3, L)$ be an element such that the apartment $g_{\mathbf{v}}(A)$ contains the vertex $\mathbf{v}$. Let $g^*_{\mathbf{v}} z_0, g^*_{\mathbf{v}} z_1, g^*_{\mathbf{v}} z_2$ be the coordinates of $\mathbb{P}^{\vee} \cong \mathbb{P}^2_L$ such that the torus S belonging to the apartment $g_{\mathbf{v}}(A) \subset \mathcal{B}$ acts diagonally. The elements $g_{\mathbf{v}}$ are choosen in a Γ-invariant way.

We define $\mathcal{Y}_{\mathbf{v}} := \{z \in Y^{s\vee} \mid d_{\mathcal{B}}(\psi^{\vee}_{\mathcal{B}}(z), \mathbf{v}) < 1,\ g^*_{\mathbf{v}} z_0 = 0\}$. Then $\mathcal{Y}_{\mathbf{v}} \cong \{z \in \Omega_1 \mid d_{\mathbf{b}}(\psi^{\vee}_{\mathbf{b}}(z), \mathbf{v}) < 1\}$. Here the space $\Omega_1 \subset \mathbb{P}^1_L$, where the $\mathbb{P}^1_L \subset \mathbb{P}^{\vee}$ is given by $g^*_{\mathbf{v}} z_0 = 0$ and $\mathbf{b}$ is the $SU(2, L)$-building that contains the apartment $g_{\mathbf{v}}(A)$.

For an edge $\mathbf{e} \notin |\mathcal{T}|$ we define two spaces $\mathcal{Y}_{\mathbf{e}}$ and $\mathcal{Y}^{\vee}_{\mathbf{e}}$. Let $\mathbf{v}, \mathbf{v}' \in \mathbf{e}$ be the vertices of type $\tau(\mathbf{v}) = 1$ and $\tau(\mathbf{v}') = 0$. Let $\mathcal{Y}_{\mathbf{e}} := \{x \in \mathcal{Y}_{\mathbf{v}'} \mid \psi_{\mathcal{B}}(x) \in \mathbf{e}, \psi_{\mathcal{B}}(x) \neq \mathbf{v}'\} \subset Y^s$. Let $\mathcal{Y}^{\vee}_{\mathbf{e}} := \{z \in \mathcal{Y}_{\mathbf{v}} \mid \psi_{\mathcal{B}}(z) \in \mathbf{e}, \psi^{\vee}_{\mathcal{B}}(z) \neq \mathbf{v}\} \subset Y^{s\vee}$.

10.11 Lemma. *Let $\mathbf{e} \in \mathcal{B}$ be an edge that is not contained in $|\mathcal{T}| = \bigcup_{\mathbf{b} \in \mathcal{T}} \mathbf{b}$. Then $\mathcal{Y}_{\mathbf{e}} \cong \mathcal{Y}^{\vee}_{\mathbf{e}}$. The isomorphism is given by taking $g^*_{\mathbf{v}} x_1 g^*_{\mathbf{v}} z_2 + g^*_{\mathbf{v}} x_2 g^*_{\mathbf{v}} z_1 = 0$, where the coordinates are such that $g^*_{\mathbf{v}} z_0 = 0$ defines the $\mathbb{P}^1_L \subset \mathbb{P}^{\vee}$ used to define $\mathcal{Y}_{\mathbf{v}} \subset Y^{s\vee}$ for the vertex $\mathbf{v} \in \mathbf{e}$ of type $\tau(\mathbf{v}) = 1$.*

Proof. Let $x \in \mathcal{Y}_{\mathbf{e}}$. Then $\psi_{\mathcal{B}}(x) \in \mathbf{e}$. Let $a \in Iso(\mathbf{v}', \mathbf{b})$ be an isotropic point, such that the conjugate $\overline{a}$ of a is contained in the $\mathbb{P}^1_L$ defined by $g^*_{\mathbf{v}} z_0 = 0$. Such an isotropic point a exists and is determined uniquely modulo π. Then $1 > |\frac{h(x,a)}{x_0 a_0}| > |\pi|$.

For each value of $(g^*_{\mathbf{v}} x_1, g^*_{\mathbf{v}} x_2)$ there are exactly $q+1$ points $(g^*_{\mathbf{v}} x_0, g^*_{\mathbf{v}} x_1, g^*_{\mathbf{v}} x_2)$ such that $b_{\mathbf{v}}(x) = 0$. For exactly one of these points $|\frac{h(x,a)}{x_0 a_0}| < 1$ holds. Therefore identifying $(g^*_{\mathbf{v}} x_0, g^*_{\mathbf{v}} x_1, g^*_{\mathbf{v}} x_2)$ with $(0, g^*_{\mathbf{v}} z_1, g^*_{\mathbf{v}} z_2)$ via $g^*_{\mathbf{v}} x_1 g^*_{\mathbf{v}} z_2 + g^*_{\mathbf{v}} x_2 g^*_{\mathbf{v}} z_1 = 0$ gives an isomorphism $\mathcal{Y}_{\mathbf{e}} \cong \mathcal{Y}^{\vee}_{\mathbf{e}}$ as stated in the lemma. □

10.12 Theorem. *Let $\mathcal{Y} := \bigcup_{\mathbf{v} \in \mathcal{B}} \mathcal{Y}_{\mathbf{v}} / \sim$, where $\sim$ is the equivalence relation obtained by applying the isomorphisms $\mathcal{Y}_{\mathbf{e}} \cong \mathcal{Y}^{\vee}_{\mathbf{e}}$ for all edges $\mathbf{e} \in \mathcal{B}$. Then the following statements hold:*

- *i) The space $\mathcal{Y}$ is a well-defined rigid analytic variety.*
- *ii) The group Γ acts discretely on the rigid analytical space $\mathcal{Y}$ and the quotient $\mathcal{Y}/\Gamma$ is a proper algebraic curve.*

Proof. It is clear from the construction that $\mathcal{Y}$ is a well-defined rigid analytic space. The group Γ acts on $\mathcal{Y}$ by permuting the admissable subspaces $\mathcal{Y}_{\mathbf{v}} \subset \mathcal{Y}$. Since Γ acts discretely on the building, it acts discretely on the space $\mathcal{Y}$.

Let us now prove that the quotient $\mathcal{Y}/\Gamma$ is proper. To do this we cover $\mathcal{Y}$ with two admissable Γ-invariant affinoid coverings $\{X_{\mathbf{v}}(r_1) \mid \mathbf{v} \in \mathcal{B}\}$ and $\{X_{\mathbf{v}}(r_2) \mid \mathbf{v} \in \mathcal{B}\}$ such that $X_{\mathbf{v}}(r_1) \subset\subset X_{\mathbf{v}}(r_2)$ for all vertices $\mathbf{v} \in \mathcal{B}$.

Let $X_{\mathbf{v}}(R) \subset \mathcal{Y}_{\mathbf{v}}$ be the analytical subspace $X_{\mathbf{v}}(R) := \{x \in \mathcal{Y}_{\mathbf{v}} \mid d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v}) \leq R\}$ if $\tau(\mathbf{v}) = 0$ and $X_{\mathbf{v}}(R) := \{z \in \mathcal{Y}_{\mathbf{v}} \mid d_{\mathcal{B}}(\psi^{\vee}_{\mathcal{B}}(z), \mathbf{v}) \leq R\}$ if $\tau(\mathbf{v}) = 1$. Here $R \in \mathbb{Q}$ is such that $0 \leq R < 1$. One verifies that the analytical space $X_{\mathbf{v}}(R)$ is in fact an affinoid subspace of $\mathcal{Y}_{\mathbf{v}}$. If $0 \leq R < R' < 1$, then $X_{\mathbf{v}}(R) \subset\subset X_{\mathbf{v}}(R')$ holds. If $1 > R \geq 1/2$, then the union $\bigcup_{\mathbf{v}\in\mathcal{B}} X_{\mathbf{v}}(R) = \mathcal{Y}$. By construction the covering is Γ-invariant. Therefore we can take r_1 and r_2 such that $1/2 \leq r_1 < r_2 < 1$ holds to obtain our admissable coverings.

After taking, if necessary, a suitable subgroup of Γ of finite index, we may assume that the action of Γ on the affinoids is such that $\gamma(X_{\mathbf{v}}(r_i)) \bigcap X_{\mathbf{v}}(r_i) = \emptyset$, $i = 1, 2$ holds for all $\gamma \in \Gamma$, $\gamma \neq 1$ and for all $\mathbf{v} \in \mathcal{B}$. Then the quotient $\mathcal{Y}/\Gamma$ is covered by the image of finitely many affinoids $X_{\mathbf{v}_i}(r_1)$ and $X_{\mathbf{v}_i}(r_2)$ with $i = 1, \ldots, s$ for some $s \geq 1$. From this the properness of the quotient follows. Since $\mathcal{Y}/\Gamma$ is a curve it is algebraic. □

10.13 Corollary. *The following two statements hold:*

i) The group $\Gamma \subset SU(3, L)$ acts linearly on $\bigcup_{\mathbf{v}\in\mathcal{B},\tau(\mathbf{v})=0} \mathcal{Y}_{\mathbf{v}}$ through the coordinates x_i, $i = 0, 1, 2$.

ii) The group $\Gamma \subset SU(3, L)$ acts linearly on $\bigcup_{\mathbf{v}\in\mathcal{B},\tau(\mathbf{v})=1} \mathcal{Y}_{\mathbf{v}}$ through the coordinates z_i, $i = 0, 1, 2$.

Proof. Clear from the construction. □

10.14. A map $\psi^{\mathcal{Y}}_{\mathcal{B}} : \mathcal{Y} \to \mathcal{B}$. Let us define a map $\psi^{\mathcal{Y}}_{\mathcal{B}} : \mathcal{Y} \to \mathcal{B}$ by:

$$\psi^{\mathcal{Y}}_{\mathcal{B}} = \begin{cases} \psi_{\mathcal{B}} & \text{on } \bigcup_{\mathbf{v}\in\mathcal{B},\tau(\mathbf{v})=0} \mathcal{Y}_{\mathbf{v}} \\ \psi^{\vee}_{\mathcal{B}} & \text{on } \bigcup_{\mathbf{v}\in\mathcal{B},\tau(\mathbf{v})=1} \mathcal{Y}_{\mathbf{v}}. \end{cases}$$

We again do not distinguish between the coordinates x_i and z_i, $i = 0, 1, 2$ for the map $\psi^{\mathcal{Y}}_{\mathcal{B}}$. In the proposition below we show that the map is well-defined.

10.15 Proposition. *The following statements hold:*

i) The map $\psi^{\mathcal{Y}}_{\mathcal{B}}: \mathcal{Y} \to \mathcal{B}$ *is well-defined and* Γ*-equivariant.*

ii) $\psi^{\mathcal{Y}}_{\mathcal{B}}|_{\mathcal{Y}^\circ} = \psi^{\mathcal{Y}^\circ}_{\mathcal{B}}$.

iii) The complement of $\mathcal{Y}^\circ$ *in* $\mathcal{Y}$ *is* $\mathcal{Y} - \mathcal{Y}^\circ = \{x \in \mathcal{Y} \mid d_{\mathcal{B}}(\psi^{\mathcal{Y}}_{\mathcal{B}}(x), |\mathcal{T}|) = 1\}$ $= \{x \in \mathcal{Y} \mid \exists (\mathbf{v} \in \mathcal{B} - |\mathcal{T}|)\ \psi^{\mathcal{Y}}_{\mathcal{B}}(x) = \mathbf{v}\}$.

Proof. Clear from the definition. □

10.16. Affinoid subspaces of the space $\mathcal{Y}$. For each edge $\mathbf{e} \in \mathcal{B}$ and each vertex $\mathbf{v} \in \mathcal{B}$ affinoid subspaces $X^{\mathcal{Y}}_{\mathbf{e}}, X^{\mathcal{Y}}_{\mathbf{v}} \subset \mathcal{Y}$ are obtained by taking $X^{\mathcal{Y}}_{\mathbf{v}} := \{x \in \mathcal{Y} \mid \psi^{\mathcal{Y}}_{\mathcal{B}}(x) = \mathbf{v}\}$ and $X^{\mathcal{Y}}_{\mathbf{e}} := \{x \in \mathcal{Y} \mid \psi^{\mathcal{Y}}_{\mathcal{B}}(x) \in \mathbf{e}\}$.

10.17 Proposition. *The covering* $\{X^{\mathcal{Y}}_{\mathbf{v}}, X^{\mathcal{Y}}_{\mathbf{e}} \mid \mathbf{v}, \mathbf{e} \in \mathcal{B}\}$ *is a* Γ*-invariant pure affinoid covering of the rigid analytic space* $\mathcal{Y}$.

Proof. Clear from the construction. □

10.18 Proposition. *Let* $\mathbf{v} \in \mathcal{B}$ *be a vertex and let* $\mathbf{e} \in \mathcal{B}$ *be an edge. Then the following statements hold:*

i) If $\tau(\mathbf{v}) = 0$*, then* $X^{\mathcal{Y}}_{\mathbf{v}} \cong X^{\Sigma_{\mathbf{b}}}_{\mathbf{v}} - R^{-1}(Iso(\mathbf{v}, \mathbf{b}))$*. Here* $\mathbf{b} \in \mathcal{T}$ *is a* $SU(2, L)$*-building such that* $\mathbf{v} \in \mathbf{b}$.

ii) If $\tau(\mathbf{v}) = 1$ *and* $\mathbf{v} \in |\mathcal{T}|$*, then* $X^{\mathcal{Y}}_{\mathbf{v}} \cong X^{\Sigma_{\mathbf{b}}}_{\mathbf{v}}$*. Here* $\mathbf{b} \in \mathcal{T}$ *is the unique* $SU(2, L)$*-building such that* $\mathbf{v} \in \mathbf{b}$.

iii) If $\tau(\mathbf{v}) = 1$ *and* $\mathbf{v} \notin |\mathcal{T}|$*, then* $X^{\mathcal{Y}}_{\mathbf{v}} \cong X^{\Omega_1}_{\mathbf{v}}$.

iv) If $\mathbf{e} \in |\mathcal{T}|$*, then* $X^{\mathcal{Y}}_{\mathbf{e}} \cong X^{\Sigma_{\mathbf{b}}}_{\mathbf{e}} - R^{-1}(Iso(\mathbf{v}, \mathbf{b}))$*. Here* $\mathbf{b} \in \mathcal{T}$ *is the unique* $SU(2, L)$*-building such that* $\mathbf{e} \in \mathbf{b}$.

v) If $\mathbf{e} \notin |\mathcal{T}|$*, then* $X^{\mathcal{Y}}_{\mathbf{e}} \cong \{x \in \mathbb{P}^2_L \mid b_{\mathbf{v}}(x) = 0, (x_1, x_2) \in X^{\Omega_1}_{\mathbf{e}}\} - R^{-1}(Iso(\mathbf{v}, \mathbf{b}))$*. Here* $\mathbf{v} \in \mathbf{e}$ *is the vertex of type* $\tau(\mathbf{v}) = 0$ *and* $\mathbf{b}$ *is any* $SU(2, L)$*-building such that* $\mathbf{e} \in \mathbf{b}$.

Proof. Clear from the construction. □

10.19 Theorem. *The reduction of the uniformising space* $\mathcal{Y}$ *is as follows:*

i) To each vertex $\mathbf{v} \in |\mathcal{T}| \subset \mathcal{B}$ *corresponds a hermitian curve.*

ii) To each vertex $\mathbf{v} \in \mathcal{B} - |\mathcal{T}|$ *corresponds a projective line with on it a hermitian form.*

iii) *The components of the reduction corresponding to the vertices* $\mathbf{v}, \mathbf{v}' \in \mathcal{B}$ *intersect if and only if there exists an edge* $\mathbf{e} \ni \mathcal{B}$ *with* $\mathbf{v}, \mathbf{v}' \in \mathbf{e}$*. In that case, the point of intersection is* ℓ*-valued (and isotropic) in both.*

Proof. Follows directly from the description of the affinoids given above and is similar to the determination of the reduction of Σ in §4 . □

10.20. Some coverings of the space $\mathcal{Y}$. The complement of $\mathcal{Y}^\circ$ in $\mathcal{Y}$ is $\mathcal{Y} - \mathcal{Y}^\circ = \bigcup_{\mathbf{v} \in \mathcal{B} - |\mathcal{T}|} X_{\mathbf{v}}^{\mathcal{Y}}$ and consists of a disjoint union of affinoids. However, the covering $\{\mathcal{Y}^\circ, X_{\mathbf{v}}^{\mathcal{Y}} \mid \mathbf{v} \in \mathcal{B} - |\mathcal{T}|\}$ is not an admissable covering. The covering $\{\mathcal{Y}^\circ, \mathcal{Y}_{\mathbf{v}} \mid \mathbf{v} \in \mathcal{B} - |\mathcal{T}|\}$ is an admissable open covering.

The covering of the rigid analytic space $\mathcal{Y}$ that consists of the open admissable subspaces $\Sigma_{\mathbf{b}}^\circ$, $\mathbf{b} \in \mathcal{T}$ and the affinoids $X_{\mathbf{v}}^{\mathcal{Y}}$ with $\mathbf{v} \in \mathcal{B} - \bigcup_{\mathbf{b} \in \mathcal{T}} \mathbf{b}$ is not an admissable covering of $\mathcal{Y}$. However, the covering $\{\Sigma_{\mathbf{b}}^\circ, \mathcal{Y}_{\mathbf{v}} \mid \mathbf{b} \in \mathcal{T}, \mathbf{v} \in \mathcal{B} - \bigcup_{\mathbf{b} \in \mathcal{T}} \mathbf{b}\}$ is an admissable open covering of $\mathcal{Y}$.

The affinoid covering $\{X_{\mathbf{v}}(R) \mid \mathbf{v} \in \mathcal{B}\}$ is an admissable affinoid covering of $\mathcal{Y}$ for $1 > R \geq 1/2$. If $R = 1/2$, then the covering is a pure affinoid covering.

10.21. An example. Let $\Gamma_0 \subset SU(3, L)$ be a discrete torsionfree co-compact subgroup that admits an almost complete transversal system $\mathcal{T}_0$ that has the following properties:

i) $\mathbf{b}, \mathbf{b}' \in \mathcal{T}_0 \Rightarrow \mathbf{b} \bigcap \mathbf{b}' = \emptyset$.

ii) Γ_0 acts transitively on the $SU(2, L)$-buildings $\mathbf{b} \in \mathcal{T}_0$.

The subset $|\mathcal{T}_0| \subset \mathcal{B}$ consist of a disjoint union of $SU(2, L)$-buildings that are permuted by the group Γ_0. For each vertex $\mathbf{v} \in \mathcal{B}$ of type $\tau(\mathbf{v}) = 0$, there exists exactly one building $\mathbf{b} \in \mathcal{T}_0$ such that $\mathbf{v} \in \mathbf{b}$. Let us fix a $SU(2, L)$-building $\mathbf{b}_0 \in \mathcal{T}_0$ and let $\Gamma_{\mathbf{b}_0} \subset \Gamma_0$ be the stabiliser of $\mathbf{b}_0$.

Then $\mathcal{Y}^\circ/\Gamma_0 \subset \mathcal{Y}/\Gamma_0$ is an open admissable subspace. We also have the isomorphism $\mathcal{Y}^\circ/\Gamma_0 \cong \Sigma_{\mathbf{b}_0}^\circ/\Gamma_{\mathbf{b}_0}$. Therefore $\mathcal{Y}^\circ/\Gamma_0 \subset \Sigma_{\mathbf{b}_0}/\Gamma_{\mathbf{b}_0}$ is again an open admissable subspace. This shows that in this case we have two distinct compactifications $\mathcal{Y}/\Gamma_0$ and $\Sigma_{\mathbf{b}_0}/\Gamma_{\mathbf{b}_0}$ of the analytical space $\Sigma_{\mathbf{b}_0}^\circ/\Gamma_{\mathbf{b}_0}$.

We will now generalise this example to connected components of almost complete transversal systems.

10.22. Connected components of $|\mathcal{T}|$. Let $\Gamma \subset SU(3, L)$ be discrete co-compact torsionfree subgroup that admits an almost complete transversal system $\mathcal{T}$. Let $T^\dagger \subset |\mathcal{T}|$ be a connected component and let $\mathcal{T}^\dagger := \{\mathbf{b} \in \mathcal{T} \mid \mathbf{b} \subset T^\dagger\}$. Let $\Gamma^\dagger$ be the stabiliser $\Gamma^\dagger := \{\gamma \in \Gamma \mid \gamma(T^\dagger) = T^\dagger\}$.

The subgroup $\Gamma^\dagger \subset \Gamma$ acts on the analytical space $\mathcal{Y}^{\dagger\circ} := \bigcup_{\mathbf{b}\in\mathcal{T}^\dagger} \Sigma^\circ_{\mathbf{b}} \subset \mathcal{Y}^\circ$. We construct a space $\mathcal{Y}^\dagger \supset \mathcal{Y}^{\dagger\circ}$ on which the group $\Gamma^\dagger$ acts discretely such that the analytical variety $\mathcal{Y}^\dagger$ has a reduction consisting of components that are all isomorphic to hermitian curves.

Let $\mathbf{v} \in T^\dagger$ be a vertex of type $\tau(\mathbf{v}) = 0$. Let $Iso^\dagger(\mathbf{v}, \mathbf{b}) := Iso(\mathbf{v}, \mathbf{b}) - \bigcap_{\mathbf{v}\in\mathbf{b},\mathbf{b}\in\mathcal{T}^\dagger} Iso(\mathbf{v}, \mathbf{b})$. Let $\mathcal{Y}^\dagger_{\mathbf{v}} := \Sigma_{\mathbf{v},\mathbf{b}} - \bigcup_{a\in Iso^\dagger(\mathbf{v},\mathbf{b})} B(a, |\pi|)$. Then $\mathcal{Y}_{\mathbf{v}} \subset \mathcal{Y}^\dagger_{\mathbf{v}}$ is an open admissable subset. The complement $\mathcal{Y}^\dagger_{\mathbf{v}} - \mathcal{Y}_{\mathbf{v}}$ consists of a ball $B(a, |\pi|)$ for each edge $\mathbf{e} \ni \mathbf{v}$, $\mathbf{e} \in \mathcal{B}$ that is not contained in $T^\dagger$.

For vertices $\mathbf{v} \in T^\dagger$ of type $\tau(\mathbf{v}) = 1$, we take $\mathcal{Y}^\dagger_{\mathbf{v}} := \mathcal{Y}_{\mathbf{v}}$. For edges $\mathbf{e} \in T^\dagger$ we take $\mathcal{Y}^\dagger_{\mathbf{e}} = \mathcal{Y}_{\mathbf{e}}$ and $\mathcal{Y}^{\dagger\vee}_{\mathbf{e}} := \mathcal{Y}^\vee_{\mathbf{e}}$. One can now identify the spaces $Y^\dagger_{\mathbf{e}} \cong \mathcal{Y}^{\dagger\vee}_{\mathbf{e}}$ for edges $\mathbf{e} \in T^\dagger$ as usual and obtain a space $\mathcal{Y}^\dagger$. The following holds:

10.23 Proposition. *Let $\mathcal{Y}^\dagger := \bigcup_{\mathbf{v}\in T^\dagger} \mathcal{Y}^\dagger_{\mathbf{v}} / \sim$, where $\sim$ denotes the equivalence relation obtained by identifying $\mathcal{Y}^\dagger_{\mathbf{e}} \cong \mathcal{Y}^{\dagger\vee}_{\mathbf{e}}$ for all edges $\mathbf{e} \in T^\dagger$. Then the following statements hold:*

i) The space $\mathcal{Y}^\dagger$ is a well defined rigid analytic variety.

ii) The group $\Gamma^\dagger$ acts discretely on $\mathcal{Y}^\dagger$ and the quotient $\mathcal{Y}^\dagger/\Gamma^\dagger$ is proper.

iii) The space $\mathcal{Y}^\dagger$ has a reduction that consists of a hermitian curve for each vertex $\mathbf{v} \in T^\dagger$.

Proof. The proof of the statements of the proposition is in general similar to those of analogous statements for the space $\mathcal{Y}$. Therefore we leave them to the reader. There is however one difference in the proofs that we will treat here. Because we have not constructed a $\Gamma^\dagger$-equivariant map from the space $\mathcal{Y}^\dagger$ to the tree $T^\dagger$ it is not obvious that the intersections $\mathcal{Y}^\dagger_{\mathbf{v}} \bigcap \mathcal{Y}^\dagger_{\mathbf{v}'}$ are empty for different vertices $\mathbf{v}, \mathbf{v}' \in T^\dagger$ of type $\tau(\mathbf{v}) = 0$.

To prove that the intersection is empty, we assume that $x \in \mathcal{Y}^\dagger_{\mathbf{v}} \bigcap \mathcal{Y}^\dagger_{\mathbf{v}'}$ and derive a contradiction. Then either $x \notin Y^s$ or $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), T^\dagger) \geq 1$. Therefore $x \in B(a, |\pi|)$ for some $a \in Iso^\dagger(\mathbf{v}, \mathbf{b})$ and $x \in B(a', |\pi|)$ for some $a' \in Iso^\dagger(\mathbf{v}', \mathbf{b}')$. These balls correspond to edges $\mathbf{e} \ni \mathbf{v}$ and $\mathbf{e}' \ni \mathbf{v}'$ that are not contained in the tree $T^\dagger$.

If $x \notin Y^s$, then the point x corresponds to an end E of the building $\mathcal{B}$. We therefore have a halfapartment H that starts in the vertex $\mathbf{v}$ and contains the edge $\mathbf{e}$ that corresponds to the end E. We also have a halfapartment H'

that starts in $\mathbf{v}'$ and contains $\mathbf{e}'$ that corresponds to the end E. Since we also have a path from $\mathbf{v}$ to $\mathbf{v}'$ that is contained in the tree $T^\dagger$, we can construct a closed path in the building $\mathcal{B}$. This cannot be. Therefore $x \in Y^s$ must hold.

Let us now assume that $x \in Y^s$ and that $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), T^\dagger) \geq 1$. Then the shortest path from $\mathbf{v}$ to $\psi_{\mathcal{B}}(x)$ goes through the edge $\mathbf{e}$. The shortest path from the vertex $\mathbf{v}'$ to $\psi_{\mathcal{B}}(x)$ goes through the edge $\mathbf{e}'$. Together with the path from $\mathbf{v}$ to $\mathbf{v}'$ contained in the tree $T^\dagger$, this again gives a closed path in the building $\mathcal{B}$. This cannot be. Hence the intersection $\mathcal{Y}^\dagger_{\mathbf{v}} \bigcap \mathcal{Y}^\dagger_{\mathbf{v}'}$ is empty. $\square$

10.24. Quotients in general. Each connected component of the analytical space $\mathcal{Y}^\circ/\Gamma$ corresponds to a Γ-orbit of a connected component $T^\dagger$ of $|\mathcal{T}|$. The component that corresponds to the Γ-orbit of $T^\dagger$ is isomorphic to an open subspace of $\mathcal{Y}^{\dagger\circ}/\Gamma^\dagger$. The different connected components of $\mathcal{Y}^\circ/\Gamma$ are glued together by adding some open admissable subspaces of Ω_1 to obtain the rigid analytic variety $\mathcal{Y}/\Gamma$.

In particular, if the group Γ acts transitively on the connected components of $|\mathcal{T}|$, then $\mathcal{Y}^\circ/\Gamma \cong \mathcal{Y}^{\dagger\circ}/\Gamma^\dagger$ is an open subspace of the two proper algebraic curves $\mathcal{Y}/\Gamma$ and $\mathcal{Y}^\dagger/\Gamma^\dagger$.

11 On generalising the construction to other groups

In this section I describe how one might generalise the construction to other algebraic groups. This provides some context for the construction of the uniformising space for arithmetic subgroups of $SU(3, L)$. The ideas presented here are based on a single example. They are bound to contain some mistakes. However, I consider it worthwile to have some idea, however faulty, of how p-adic uniformisation of algebraic varieties in general might work.

Our scenario is as follows. Let H be a K-split semisimple group defined over K, such that all the roots in the root sytem Φ_H of H have the same length. We assume that there exists a rigid analytic variety $\mathcal{Y}_H$ on which the group $H(K)$ acts and such that the quotients $\mathcal{Y}_H/\Gamma_H$ are proper algebraic varieties for discrete co-compact subgroups $\Gamma_H \subset H(K)$. The existence of such a space $\mathcal{Y}_H$ is only known for groups H such that the root system Φ_H is of type A. In those cases the space $\mathcal{Y}_H$ equals Drinfel'ds symmetric space for the group $H(K)$.

Let G be a quasisplit group defined over K with root sytem Φ_G. We assume that the root sytem Φ_G contains roots of different length. Let the long roots of Φ_G form a root system of type Φ_H. For the group $G(K)$ no suitable rigid space $\mathcal{Y}_G$ exists, on which $G(K)$ acts and such that the quotients $\mathcal{Y}_G/\Gamma$ are proper algebraic varieties for all discrete co-compact subgroups $\Gamma \subset G(K)$. However, using étale coverings of the spaces $\mathcal{Y}_H$ and coverings of the affine building $\mathcal{B}_G$ of $G(K)$ by $H(K)$-subbuildings, we expect that one can construct for an arithmetic discrete co-compact subgroup $\Gamma \subset G(K)$ a rigid analytic variety $\mathcal{Y}_\Gamma$ on which Γ acts discretely with proper algebraic quotients. The reduction of the algebraic varieties uniformised by these spaces consists of components that are most likely compactifications of Deligne-Lusztig varieties belonging to a Coxeter element in the Weyl group of a semisimple group.

Let us give a brief outline of this section. We define almost complete transversal systems for semisimple algebraic groups G defined over K. We sketch how one might construct Γ-invariant ones for discrete co-compact arithmetic subgroups $\Gamma \subset G(K)$. We describe the analytic spaces on which we expect the arithmetic group Γ to act discretely. These analytic spaces are closely connected to Deligne-Lusztig varieties for the finite group $G(k)$, where k denotes the residue field of K. Finally, we compare our construction with real symmetric spaces and Shimura varieties.

11.1. Groups and buildings. Let us recall some definitions that are used in the description of the transversal systems. Let G be an absolutely almost simple group defined over K of K-rank $rk_K(G) = r$, with root system Φ_G and affine building $\mathcal{B}_G$.

Let $T \subset G$ be a maximal K-split torus. The characters with which the torus T acts on the T-invariant additive subgroups of G correspond to the roots $\alpha \in \Phi_G$. Let $N \subset G$ be the normaliser of the torus T. The *Weyl group* of the torus is the group $W := N(K)/T(K)$. The group W is also the Weyl group of the root sytem Φ_G. The *affine Weyl group* W_{aff} is defined as $W_{aff} := N(K)/T(K^\circ)$.

Let $A \subset \mathcal{B}_G$ be the apartment corresponding to the maximal K-split torus $T \subset G$. As a topological space the apartment A is isomorphic to $\mathbb{R}^r$, where r is the K-rank of G. The torus T acts on A by translations. On the apartment A one defines hyperplanes using the roots $\alpha \in \Phi_G$. For each $n \in \mathbb{Z}$ and each $\alpha \in \Phi_G$ one has a hyperplane defined as follows: $\{z \in A \mid (z, \alpha) = n\}$. These hyperplanes are the *walls* of the apartment. The polyhedral decomposition coming from these walls give the apartment A a polysimplicial structure. The affine Weyl group is generated by the reflections in the walls of the apartment A.

There are $r+1$ types of vertices in the apartment A and $\mathcal{B}_G$. The type $\tau(\mathbf{v})$ of the vertex $\mathbf{v} \in A$ will be denoted by an integer $0, \ldots, r$. A vertex $\mathbf{v} \in \mathcal{B}_G$ is called *special* if the stabiliser of $\mathbf{v}$ in the affine Weyl group of an apartment $A \ni \mathbf{v}$ equals the Weyl group of the root system Φ_G. Our numbering of the types of vertices is such that vertices of $\tau(\mathbf{v}) = 0$ are special.

A *flat* $F \subset \mathcal{B}_G$ is a subset of the building $\mathcal{B}_G$ that is an affine linear subspace of an apartment $A \subset \mathcal{B}_G$. Therefore F is a translation inside A of a linear subspace $\mathbb{R}^s \subset A \cong \mathbb{R}^r$ with $0 \leq s \leq r$. We call a flat *simplicial* if it is the union of polysimplices $\sigma \in A$. A simplicial flat equals the intersection of a finite number (possibly zero) of walls in an apartment A.

A polysimplicial subcomplex $P \subset \mathcal{B}_G$ is called *reductive* if it is isomorphic to the product of a simplicial flat $F \subset \mathcal{B}_G$ and a subbuilding $\mathcal{B}_{G'} \subset \mathcal{B}_G$. Here $G'(K) \subset G(K)$ is a semisimple subgroup. The stabiliser of a reductive simplicial subcomplex of the building $\mathcal{B}_G$ is a reductive subgroup of $G(K)$.

11.2. Transversal systems. We want to cover the building $\mathcal{B}_G$ by subbuildings that are all isomorphic to the building of a suitable subgroup $H \subset G$. To have a useful covering we need to impose conditions on the group H, conditions on how these subbuildings intersect and what type of vertices they contain. We will assume that the group G is almost simple. Then the root system Φ_G is irreducible.

An *almost complete transversal system* $\mathcal{T}$ in $\mathcal{B}_G$ is a set of subbuildings $\mathbf{b} \subset \mathcal{B}_G$ such that the following conditions hold:

i) All buildings $\mathbf{b} \in \mathcal{T}$ are isomorphic to the building $\mathbf{b}_H$ of a semisimple group $H(K) \subset G(K)$ of $rk_K(H) = rk_K(G) = r$.

ii) The group H is quasisplit over K, i.e. the group H contains a Borel subgroup that is defined over the field K. Therefore the group H splits over an unramified extension of K.

iii) All roots $\alpha \in \Phi_H$ have the same length.

iv) If $g \in G(K)$ and $\mathbf{b} \in \mathcal{T}$ are such that the intersection $g(\mathbf{b}) \bigcap \mathbf{b}$ contains an apartment, then $g(\mathbf{b}) = \mathbf{b}$.

v) For $\mathbf{b}, \mathbf{b}' \in \mathcal{T}$, $\mathbf{b} \neq \mathbf{b}'$, the intersection $\mathbf{b} \bigcap \mathbf{b}'$ is either empty or a reductive simplicial complex of codimension at least one.

vi) If $\mathbf{v} \in \mathcal{B}_G$ is a vertex of type $\tau(\mathbf{v}) = 0$, then there exists a subbuilding $\mathbf{b} \in \mathcal{T}$ such that $\mathbf{v} \in \mathbf{b}$. (Vertices $\mathbf{v}$ of type $\tau(\mathbf{v}) = 0$ are special vertices.)

Given a group G, conditions (i)-(iv) determine the group H more or less uniquely. Since the group H is quasisplit and all roots $\alpha \in \Phi_H$ have the same length, the group is K-split of type A, D or E. The fourth condition implies that the group H is invariant under the action of the Weyl group of G. It follows that if the group G is quasisplit, then the subgroup $H \subset G$ is such that Φ_H consists of the longest roots of Φ_G. In general the group H is the group determined by the longest roots $\alpha \in \Phi_G$, such that the root groups U_α contain non-trivial elements defined over a non-ramified extension of K. Therefore Φ_H consists of the long roots of a maximal quasisplit subgroup $G^u \subset G$.

The intersection of buildings $\mathbf{b}, \mathbf{b}' \in \mathcal{T}$ is of codimension at least one by condition (v). If the group G is quasisplit, then the intersection is again a building. If the group splits over an extension of K that is ramified, then I expect that also (intersections of) walls occur as intersections $\mathbf{b} \bigcap \mathbf{b}'$. These intersections are not buildings.

Condition (vi) ensures that the complement of the subset $|\mathcal{T}| := \bigcup_{\mathbf{b} \in \mathcal{T}} \mathbf{b} \subset \mathcal{B}_G$ is of lower dimension than the building itself. Every wall contained in an apartment $A \subset \mathcal{B}_G$ contains a vertex $\mathbf{v}$ of type $\tau(\mathbf{v}) = 0$. Indeed, the fact that the stabiliser in the affine Weyl group W_{aff} of the vertex $\mathbf{v}$ is the Weyl group of Φ_G implies that the vertex $\mathbf{v}$ is the intersection of a set of walls that are in one to one correspondence to the roots $\alpha \in \Phi_G$.

11.3. The intersections of subbuildings. Let $\Gamma \subset G(K)$ be an arithmetic discrete co-compact subgroup. Let $\mathbf{b}, \mathbf{b}' \subset \mathcal{B}_G$ be the subbuildings belonging to subgroups $H_{\mathbf{b}}(K) \cong H(K)$ and $H_{\mathbf{b}'}(K) \cong H(K)$, respectively. We assume that the intersections $\Gamma \bigcap H_{\mathbf{b}}(K)$ and $\Gamma \bigcap H_{\mathbf{b}'}(K)$ are again co-compact.

Let us assume that the intersection $\mathbf{b} \bigcap \mathbf{b}'$ is not compact. Let $J \subset \mathbf{b} \bigcap \mathbf{b}'$ be a rational half line and let $\sigma \in \mathbf{b} \bigcap \mathbf{b}'$ be a simplex such that the intersection $\sigma \bigcap J$ is non-empty. Then there exists an element $\gamma \in \Gamma \bigcap H_{\mathbf{b}}(K)$ such that $\gamma^m(\sigma) \bigcap J \neq \emptyset$ for all $m \geq 0$. The group $\Gamma \bigcap H_{\mathbf{b}'}(K)$ contains an element γ' with the same property. Since the group Γ is discrete, an equality $(\gamma')^{n_1} = \gamma^{n_2}$ must hold for suitable $n_1, n_2 > 0$. It follows that the intersection

$\mathbf{b} \bigcap \mathbf{b}'$ contains the simplices $\gamma^i(\sigma)$ for all $i \in \mathbb{Z}$. Therefore the intersection $\mathbf{b} \bigcap \mathbf{b}'$ contains a line that extends the half line J. Since this holds for any half line in the intersection $\mathbf{b} \bigcap \mathbf{b}'$, the intersection equals the product of a compact set with a flat and possibly a building.

Our condition on the intersections in a transversal system amounts to choosing subbuildings such that the compact part of the intersections is trivial, i.e. a vertex. Since the intersection $\mathbf{b} \bigcap \mathbf{b}'$ is simplicial, it follows that the intersection $\mathbf{b} \bigcap \mathbf{b}'$ is a reductive polysimplicial subcomplex of the building.

11.4. Long roots and subgroups. For the convenience of the reader we provide the root systems $\Phi_H \subset \Phi_G$ that occur. The irreducible root systems in which all the roots have the same length are the root systems $\Phi_G = \Phi_H$ of type $A_r, r > 0$; $D_r, r > 3$; $E_6; E_7$ and E_8. Let Φ_G be an irreducible root system that contains roots of different length. The pairs (Φ_G, Φ_H) that can occur are as follows:

(B_3, A_3) ; (B_r, D_r), $r > 3$;

(BC_r, A_1^r), $r > 0$;

(C_r, A_1^r), $r > 1$; (note $B_2 \cong C_2$)

(F_4, D_4);

(G_2, A_2).

One notes that the root system $\Phi_H \subset \Phi_G$ is either of type A or of type D for the root systems Φ_G that contain roots of unequal lengths.

11.5. The complement of an almost complete transversal system. Let us look in some detail at the complement of the union $|\mathcal{T}| = \bigcup_{\mathbf{b} \in \mathcal{T}} \mathbf{b}$ of the set of subbuildings in $\mathcal{B}_G$.

Let $B_1(\mathcal{T})$ denote the set of simplices $\sigma \in \mathcal{B}_G$ that have empty intersection with the subset $|\mathcal{T}| \subset \mathcal{B}_G$. Then $B_1(\mathcal{T}) = \{\sigma \in \mathcal{B}_G \mid \forall(\mathbf{b} \in \mathcal{T})\ \sigma \bigcap \mathbf{b} = \emptyset\}$ is the complement in $\mathcal{B}_G$ of $\mathcal{T}$. We call the union of the simplices $\sigma \in \mathcal{B}_G$ such that $\sigma \bigcap \bigcup_{\mathbf{b} \in \mathcal{T}} \mathbf{b} = \emptyset$ the *boundary* $|B_1(\mathcal{T})| \subset \mathcal{B}_G$ of the transversal system $\mathcal{T}$. One would like to cover the boundary $|B_1(\mathcal{T})|$ by buildings. It turns out that the boundary, as a subset of the building $\mathcal{B}_G$, in general behaves badly and cannot be covered by buildings. However, one can substantially improve the behaviour of the boundary by subdividing some of the simplices that are not contained in $|\mathcal{T}|$.

Let us look at this in more detail. Let $\mathcal{T}$ be an almost complete transversal system of $H(K)$-subbuildings $\mathbf{b} \subset \mathcal{B}_G$. Let $\mathbf{b}, \mathbf{b}' \in \mathcal{T}$ be such that the intersection $\mathbf{b} \bigcap \mathbf{b}'$ is empty. We assume that there exists an apartment $A \subset \mathcal{B}_G$ such that $A \bigcap |\mathcal{T}| = (A \bigcap \mathbf{b}) \bigcup (A \bigcap \mathbf{b}')$ holds. Then the intersections $A \bigcap \mathbf{b}$ and $A \bigcap \mathbf{b}'$ together contain all vertices $\mathbf{v} \in A$ of type $\tau(\mathbf{v}) = 0$. Convexity of the intersections then implies that both $A \bigcap \mathbf{b}$ and $A \bigcap \mathbf{b}'$ are half apartments. These two half apartments are bounded by neighbouring parallel walls in A. The walls are defined by some short root $\alpha \in \Phi_G$. Indeed, if α were a long root, the walls defined by α would actually be walls inside $\mathbf{b}$ or $\mathbf{b}'$. Therefore the walls belonging to long roots cannot define a boundary of $\mathbf{b} \bigcap A$ or $\mathbf{b}' \bigcap A$.

The intersection $|B_1(\mathcal{T})| \bigcap A$ consists of the simplices $\sigma \in A$ that are between the two walls bounding $\mathbf{b} \bigcap A$ and $\mathbf{b}' \bigcap A$ and, moreover, do not intersect these walls. In particular, if the root system Φ_G is not of type BC_r, then these simplices $\sigma \in |B_1(\mathcal{T})| \bigcap A$ do not form a wall, since there is no wall between these two parallel walls. The subspace $A - (|B_1(\mathcal{T})| \bigcap A) \subset A$ is either connected or it consists of two connected components. If $A - (|B_1(\mathcal{T})| \bigcap A)$ consists of two connected components, the intersection $|B_1(\mathcal{T})| \bigcap A \subset A$ is a connected subspace of codimension one. Moreover, it separates the two half apartments $\mathbf{b} \bigcap A$ and $\mathbf{b}' \bigcap A$ from each other.

If the subspace $A - (|B_1(\mathcal{T})| \bigcap A) \subset A$ is connected, then the subspace $|B_1(\mathcal{T})| \bigcap A \subset A$ does not separate the half apartments $\mathbf{b} \bigcap A$ and $\mathbf{b}' \bigcap A$ from each other. In this case one can subdivide the chambers that are not contained in $(\mathbf{b}' \bigcap A) \bigcup (\mathbf{b} \bigcap A)$ in such a way that the simplices $\sigma \in A$ (in the refined simplicial structure of the apartment A) that have empty intersection with both $\mathbf{b}$ and $\mathbf{b}'$ form a connected subspace of the apartment A that separates the half apartments $\mathbf{b} \bigcap A$ and $\mathbf{b}' \bigcap A$ from each other. In the apartment with this refined simplicial structure one obtains a modified boundary $|B_1(\mathcal{T})| \bigcap A$ that is such that $|B_1(\mathcal{T})| \bigcap A$ forms a connected boundary of codimension one in A, that does separate the two half apartments $\mathbf{b} \bigcap A$ and $\mathbf{b}' \bigcap A$ from each other.

After such a subdivision of simplices of the apartment A, if necessary, we may assume that the half apartments $\mathbf{b} \bigcap A$ and $\mathbf{b}' \bigcap A$ are separated by a connected subspace $|B_1(\mathcal{T})| \bigcap A \subset A$ of codimension one. We may assume that the torus $T(K) \bigcap H_{\mathbf{b}} \bigcap H_{\mathbf{b}'}$ acts on the space $|B_1(\mathcal{T})| \bigcap A$. This torus is a maximal K-split torus of the group $H_{\mathbf{b}} \bigcap H_{\mathbf{b}'}$. The subspace $|B_1(\mathcal{T})| \bigcap A \subset A$ therefore can be seen as an apartment $A_{H_{\mathbf{b}} \bigcap H_{\mathbf{b}'}}$ in the building of the group $H_{\mathbf{b}}(K) \bigcap H_{\mathbf{b}'}(K)$. Therefore the union of the $H_{\mathbf{b}}(K) \bigcap H_{\mathbf{b}'}(K)$ images of the apartment $A_{H_{\mathbf{b}} \bigcap H_{\mathbf{b}'}} := |B_1(\mathcal{T})| \bigcap A$ gives an embedding of the building of the group $H_{\mathbf{b}}(K) \bigcap H_{\mathbf{b}'}(K)$ into the building $\mathcal{B}_G$.

The thus obtained building of the group $H_{\mathbf{b}}(K) \bigcap H_{\mathbf{b}'}(K)$ is contained in the interior of the set $\{z \in \mathcal{B}_G \mid d_{\mathcal{B}_G}(z, \mathbf{b}) + d_{\mathcal{B}_G}(z, \mathbf{b}') = d_{\mathcal{B}_G}(\mathbf{b}, \mathbf{b}')\} \subset \mathcal{B}_G$ and separates the buildings $\mathbf{b}$ and $\mathbf{b}'$ from each other. Subdividing the simplices of the building $\mathcal{B}_G$ in such a way that defines the apartments of this $H_{\mathbf{b}}(K) \bigcap H_{\mathbf{b}'}(K)$-subbuilding gives an refinement of the simplicial structure of the building $\mathcal{B}_G$. One can do this for all pairs of buildings $\mathbf{b}, \mathbf{b}' \in \mathcal{T}$ such that there exists an apartment $A \subset \mathcal{B}_G$ such that $A \bigcap |\mathcal{T}| = (A \bigcap \mathbf{b}) \bigcup (A \bigcap \mathbf{b}')$ and the intersections $A \bigcap \mathbf{b}$ and $A \bigcap \mathbf{b}'$ together contain all vertices $\mathbf{v} \in A$ of type $\tau(\mathbf{v}) = 0$.

In such a building with possibly a refined simplicial structure one can propose the following condition on the boundary of the transversal system $\mathcal{T}$:

vii) The simplices $\sigma \in \mathcal{B}_G$ such that $\sigma \bigcap \bigcup_{\mathbf{b} \in \mathcal{T}} \mathbf{b} = \emptyset$ form a union of buildings $\mathcal{B}' \subset \mathcal{B}_G$ of rank $rk_K(\mathcal{B}') = rk_K(\mathcal{B}_G) - 1 = r - 1$. Here $rk_K(\mathcal{B}_G)$ denotes the K-rank of $\mathcal{B}_G$.

A transversal system is called *complete* if $|\mathcal{T}| = \mathcal{B}_G$. Then $|B_1(\mathcal{T})| = \emptyset$. Complete Γ-invariant transversal systems for discrete co-compact subgroups $\Gamma \subset G(K)$ are probably very rare. Only for small primes p and buildings of low rank they might exist.

11.6. Short roots and apartments. Root systems of type B_r and C_r. We look in some detail at the simplicial structure of the apartment A for some specific root systems Φ_G. In particular, we study the simplices between two neighbouring walls that correspond to a short root $\alpha \in \Phi_G$. This adds some detail to the description of the boundary $|B_1(\mathcal{T})|$ of a transversal system.

We call the subset $\{z \in A \mid n < (z, \alpha) < n + 1\} \subset A$ a *corridor* belonging to the root $\alpha \in \Phi_G$. The complement in the apartment A of a corridor consists of two disjoint half apartments that are bounded by two neighbouring walls of type α. We assume that the buildings $\mathbf{b}, \mathbf{b}' \in \mathcal{T}$ are $H(K)$-subbuildings such that the complement of $|\mathcal{T}| \bigcap A = (\mathbf{b} \bigcap A) \bigcup (\mathbf{b}' \bigcap A)$ is a corridor, which we will denote by $\mathcal{C}_{\mathbf{b},\mathbf{b}'}$. The corridor $\mathcal{C}_{\mathbf{b},\mathbf{b}'}$ belongs to a short root $\alpha \in \Phi_G$. The intersection $|B_1(\mathcal{T})| \bigcap A$ is contained in the corridor and equals $|B_1(\mathcal{T})| \bigcap \mathcal{C}_{\mathbf{b},\mathbf{b}'}$.

Let us consider root systems Φ_G such that the apartment A contains at least two types $\tau(\mathbf{v})$ of vertices that are special. In this case the subspace $A - |B_1(\mathcal{T})| \subset A$ is connected, as we will show below. Hence $|B_1(\mathcal{T})| \bigcap A$ does not separate the half apartments $\mathbf{b} \bigcap A$ and $\mathbf{b}' \bigcap A$ from each other.

Since a special vertex is the intersection of walls belonging to each root $\alpha \in \Phi_G$, there are no special vertices in the corridor $\mathcal{C}_{\mathbf{b},\mathbf{b}'}$ that belongs to a short root $\alpha \in \Phi_G$. If the root system Φ_G has two types of special vertices, then there is an edge $\mathbf{e} \in A$ that consists of two special vertices $\mathbf{v}$ and $\mathbf{v}'$. Let the vertex $\mathbf{v} \in \mathbf{e}$ be contained in one of the walls that bound the corridor $\mathcal{C}_{\mathbf{b},\mathbf{b}'}$. We may assume that the edge $\mathbf{e}$ itself is not contained in the wall that bounds the corridor $\mathcal{C}_{\mathbf{b},\mathbf{b}'}$. After applying the reflection in the wall, if necessary, we may assume that the interior of the edge $\mathbf{e}$ is contained in the corridor $\mathcal{C}_{\mathbf{b},\mathbf{b}'}$ between the two walls. Therefore the vertex $\mathbf{v}'$ of the edge $\mathbf{e}$ is contained in the neighbouring wall. In particular, the edge $\mathbf{e}$ joins the two half apartments $\mathbf{b} \bigcap A$ and $\mathbf{b}' \bigcap A$ together. Therefore the simplices contained in the interior of the corridor do not separate the two half apartments from each other.

Let us now describe for the specific root systems Φ_G of type C_r and B_r, $r > 1$ how to refine the simplicial structure of the apartment A in such a way that the intersection $|B_1(\mathcal{T})| \bigcap A$ separates the half apartments $\mathbf{b} \bigcap A$ and $\mathbf{b}' \bigcap A$. In both cases this can be done by adding a hyperplane to the set of walls of the apartment.

Let e_i, $i = 1, \ldots, r$ be a standard orthonormal basis of $\mathbb{R}^r$. Let $\Phi_G = \{\pm 2e_i, \pm e_i \pm e_j \mid 1 \leq i, j \leq r, i \neq j\}$ be the root system of type C_r. The vertices $\mathbf{v} \in A \cong \mathbb{R}^r$ form an integer lattice Λ_A. The lattice of vertices is the lattice $\Lambda_A = \frac{1}{2}\mathbb{Z}^r \subset A \cong \mathbb{R}^r$. Let $\alpha \in \Phi_G$ be a short root, e.g. $\alpha = e_1 - e_2$. All the vertices of the apartment contained in the corridor defined by $n < (z, \alpha) < n + 1$ satisfy the equation $(z, \alpha) = n + \frac{1}{2}$. Therefore by adding the hyperplane defined by $(z, \alpha) = n + \frac{1}{2}$ to the set of walls of the apartment A we obtain a simplicial subdivision of the apartment. Then the intersection $|B_1(\mathcal{T})| \bigcap A = \{z \in A \mid (z, \alpha) = n + \frac{1}{2}\}$ separates the half apartments $\mathbf{b} \bigcap A$ and $\mathbf{b}' \bigcap A$ from each other.

Let $\Phi_G = \{\pm e_i, \pm e_i \pm e_j \mid 1 \leq i, j \leq r, i \neq j\}$ be the root system of type B_r. In this case the lattice of vertices is the lattice $\Lambda_A = \{\frac{1}{2}n \mid n \in \mathbb{Z}^r, \sum_{i=1}^r n_i = 0 \mod 2\}$. Let $\alpha \in \Phi_G$ be a short root, e.g. $\alpha = e_1$. The vertices of the apartment contained in the corridor defined by $n < (z, \alpha) < n + 1$ satisfy the equation $(z, \alpha) = n + \frac{1}{2}$. By adding the hyperplane defined by $(z, \alpha) = n + \frac{1}{2}$ to the set of walls of the apartment A we obtain the required simplicial subdivision of the apartment. Then the intersection $|B_1(\mathcal{T})| \bigcap A = \{z \in A \mid (z, \alpha) = n + \frac{1}{2}\}$ separates the half apartments $\mathbf{b} \bigcap A$ and $\mathbf{b}' \bigcap A$ from each other.

If the root system Φ_G is of type C_r or of type B_r, then the subset $\{z \in \mathcal{B}_G \mid d_{\mathcal{B}_G}(z, \mathbf{b}) = d_{\mathcal{B}_G}(z, \mathbf{b}') = d_{\mathcal{B}_G}(\mathbf{b}, \mathbf{b}')/2\} \subset \mathcal{B}_G$ defines a subbuilding that corresponds to the group $H_{\mathbf{b}}(K) \bigcap H_{\mathbf{b}'}(K) \subset G(K)$. Here $\mathbf{b}, \mathbf{b}' \in \mathcal{T}$ are such that there exists an apartment $A \subset \mathcal{B}_G$ such that $A \bigcap |\mathcal{T}| = (\mathbf{b} \bigcap A) \bigcup (\mathbf{b}' \bigcap A)$ holds. Adding the hyperplanes to $\mathcal{B}_G$ that define the apartments of these buildings gives a refinement of the simplicial structure of the building $\mathcal{B}_G$, such that the boundary $|B_1(\mathcal{T})|$ of the transversal system $\mathcal{T}$ is covered by subbuildings.

11.7. Short roots and apartments. Root systems of type G_2. Let us now look at a root system Φ_G such that all special vertices $\mathbf{v} \in A$ are of type $\tau(\mathbf{v}) = 0$. We consider the root system $\Phi_G = G_2$. We will show that in this case $|B_1(\mathcal{T})| \bigcap A$ separates the two half apartments $\mathbf{b} \bigcap A$ and $\mathbf{b}' \bigcap A$ from each other. So in this case no subdivision of the simplicial structure of the building $\mathcal{B}_G$ is needed to obtain a good boundary of an almost complete transversal system $\mathcal{T}$.

Let us first describe in some detail the apartment A for the root system Φ_G of type G_2. There are three types $\tau(\mathbf{v})$ of vertices $\mathbf{v}$. The stabiliser of a vertex $\mathbf{v} \in A$ of type $\tau(\mathbf{v}) = 0, 1$ and 2 in the affine Weyl group is a finite Weyl group of type G_2, $A_1 \times A_1$ and A_2, respectively. A vertex $\mathbf{v} \in A$ of type $\tau(\mathbf{v}) = 0$ is the intersection of exactly six walls, three of them belong to short roots and three to long roots. A vertex $\mathbf{v}$ of type $\tau(\mathbf{v}) = 1$ is the intersection of exactly two walls, one belongs to a long root and one to a short root. A vertex of type $\tau(\mathbf{v}) = 2$ is the intersection of exactly three walls, all three belong to long roots $\alpha \in \Phi_G$. A wall belonging to a long root contains vertices of all three types, whereas a wall that belongs to a short root only contains vertices $\mathbf{v}$ of type $\tau(\mathbf{v}) = 0$ and $\tau(\mathbf{v}) = 1$.

Let us discuss the simplices that are contained in the corridor $\mathcal{C}_{\mathbf{b},\mathbf{b}'}$ between two neighbouring walls that belong to a short root $\alpha \in \Phi_G$. There are two types of vertices $\mathbf{v}$ in the corridor, the types $\tau(\mathbf{v})$ are one and two. These vertices are joined by edges. These edges are contained in walls that belong to long roots. A vertex $\mathbf{v}$ of type $\tau(\mathbf{v}) = 2$ is joined to two vertices $\mathbf{v}'$ and $\mathbf{v}''$ of type $\tau(\mathbf{v}') = \tau(\mathbf{v}'') = 1$ by edges $\mathbf{e}$ and $\mathbf{e}'$ that are contained in the corridor $\mathcal{C}_{\mathbf{b},\mathbf{b}'}$. The edges $\mathbf{e}$ and $\mathbf{e}'$ are contained in different walls belonging to long roots. These walls intersect in the vertex $\mathbf{v}$. A vertex $\mathbf{v}'$ of type

$\tau(\mathbf{v}') = 1$ is joined to two vertices $\mathbf{v}''$ and $\mathbf{v}'''$ of type $\tau(\mathbf{v}'') = \tau(\mathbf{v}''') = 2$. These two vertices $\mathbf{v}''$ and $\mathbf{v}'''$ are contained in the unique wall belonging to a long root that contains the vertex $\mathbf{v}'$. In particular, the union of the edges in the corridor $\mathcal{C}_{\mathbf{b},\mathbf{b}'}$ forms a connected codimension one subspace of the apartment A and separates both half apartments $\mathbf{b} \bigcap A$ and $\mathbf{b}' \bigcap A$ from each other.

11.8. Example: The arithmetic groups $SO(2r+1, \mathbb{Z}[1/p])$ for $r = 1, 2, 3$. Let G be the special orthogonal group $SO(n, \mathbb{Z})$ preserving the standard orthogonal form $f_n(x, y) = \sum_{i=1}^{n} x_i y_i$ on $\Lambda := \mathbb{Z}^n$. Let $p > 2$ be a prime such that -1 is a square in $\mathbb{Q}_p$. Then $p \equiv 1 \mod 4$. Let $K := \mathbb{Q}_p$. The group $G(\mathbb{Q}_p)$ is split over $\mathbb{Q}_p$ and $\Gamma := G(\mathbb{Z}[1/p]) = SO(n, \mathbb{Z}[1/p]) \subset G(\mathbb{Q}_p)$ is an arithmetic discrete co-compact subgroup.

The root system of $G(\mathbb{Q}_p)$ is of type D_r if $n = 2r$ and of type B_r if $n = 2r+1$, where we define the root system B_1 as being $B_1 := A_1$. Therefore only for odd values of $n > 2$ does the root system Φ_G contain roots of different length. The groups $H(\mathbb{Q}_p)$ that correspond to the subset of long roots $\Phi_H \subset \Phi_G$ are the groups $SO(2r, \mathbb{Q}_p)$. These have a root system Φ_H of type D_r. The group $H(\mathbb{Q}_p)$ is the stabiliser in $G(\mathbb{Q}_p)$ of an anisotropic vector $v \in \Lambda \otimes \mathbb{Q}_p$ such that $f_n(v, v) \in \mathbb{Q}_p^*$ is a square. In the case where $r = 1$, the stabiliser of such an anisotropic vector v is a maximal $\mathbb{Q}_p$-split torus $T(\mathbb{Q}_p) \subset SO(3, \mathbb{Q}_p)$. Therefore we can extend our definition of subgroups $H(\mathbb{Q}_p) \subset G(\mathbb{Q}_p)$ to the case where $r = 1$ by defining $H(\mathbb{Q}_p) := T(\mathbb{Q}_p)$ in this case.

One can describe the building $\mathcal{B}_G$ using equivalence classes of $\mathbb{Z}_p$-modules in $\mathbb{Q}_p^n$. To a vertex $\mathbf{v} \in \mathcal{B}_G$ one associates an equivalence class $[M_{\mathbf{v}}]$ of $\mathbb{Z}_p$-modules contained in $\Lambda \otimes \mathbb{Q}_p \cong \mathbb{Q}_p^n$. We will only define the equivalence classes $[M_{\mathbf{v}}]$ belonging to the vertices $\mathbf{v} \in \mathcal{B}_G$ that are special. We fix a vertex $\mathbf{v}_0$ of type $\tau(\mathbf{v}_0) = 0$ and take $M_{\mathbf{v}_0} := \Lambda \otimes \mathbb{Z}_p \cong (\mathbb{Z}_p)^n$. For a special vertex $\mathbf{v} \in \mathcal{B}_G$ there exists an element $g_{\mathbf{v}} \in G(\mathbb{Q}_p)$ such that $\mathbf{v} = g_{\mathbf{v}}(\mathbf{v}_0)$. Then $[M_{\mathbf{v}}] := [g_{\mathbf{v}}(M_{\mathbf{v}_0})]$. To the vertex $\mathbf{v}$ we associate a $\mathbb{Z}$-lattice $\Lambda_{\mathbf{v}} := \Lambda[1/p] \bigcap M_{\mathbf{v}}$. The arithmetic subgroup $\Gamma \subset G(\mathbb{Q}_p)$ acts on the set of lattices $\Lambda_{\mathbf{v}}$ belonging to the special vertices $\mathbf{v} \in \mathcal{B}_G$.

Let us now define a set $\mathcal{T}_r$ of $H(\mathbb{Q}_p)$-subbuildings (or of apartments if $r = 1$) of the building $\mathcal{B}_G$. Let $v \in \Lambda$ be a vector. The stabiliser in G of the vector $v \in \Lambda$ will be denoted by H_v. Let $\mathbf{b}_v \subset \mathcal{B}_G$ be the subbuilding (apartment if $r = 1$) stabilised by the group $H_v(\mathbb{Q}_p) \subset G(\mathbb{Q}_p)$. Let $\mathcal{T}_r$ be the set consisting of the buildings $\mathbf{b}_v$ (or apartments if $r = 1$) belonging to the groups $H_v(\mathbb{Q}_p) \subset G(\mathbb{Q}_p)$ that stabilise a vector $v \in \Lambda$ such that $f_{2r+1}(v, v) = p^{2m}$ for some $m \in \mathbb{Z}_{\geq 0}$. By construction the set $\mathcal{T}_r$ is $SO(2r+1, \mathbb{Z}[1/p])$-invariant.

It is well-known, that for $n \leq 7$ all positive definite unimodular integer lattices in $\mathbb{R}^n$ are isomorphic to $\mathbb{Z}^n$. Therefore the arithmetic group Γ acts transitively on the special vertices $\mathbf{v} \in \mathcal{B}_G$ when $n \leq 7$. These vertices $\mathbf{v}$ are of type $\tau(\mathbf{v}) = 0$ and $\tau(\mathbf{v}) = r$ for odd $n = 2r + 1$. In particular, the subset $|\mathcal{T}_r| = \bigcup_{\mathbf{b} \in \mathcal{T}_r} \mathbf{b} \subset \mathcal{B}_G$ contains all the vertices $\mathbf{v}$ of type $\tau(\mathbf{v}) = 0$ and $\tau(\mathbf{v}) = r$, if $r = 1, 2, 3$. We claim that $\mathcal{T}_r$ is an almost complete transversal system consisting of $H(\mathbb{Q}_p)$-buildings if $r = 2, 3$ and consisting of apartments if $r = 1$.

We will now show that distinct buildings contained in the set $\mathcal{T}_r$ intersect transversally. Let $v_1, v_2 \in \Lambda$ be vectors such that $f_n(v_i, v_i) = p^{2m_i}$ for $i = 1, 2$. A vertex $\mathbf{v}$ is contained in the intersection $\mathbf{b}_{v_1} \bigcap \mathbf{b}_{v_2}$ if an only if there exists a representative $M_{\mathbf{v}}$ of the equivalence class of $\mathbb{Z}_p$-modules belonging to $\mathbf{v}$ such that $v_i/p^{m_i} \in M_{\mathbf{v}}$ and the image of $v_i/p^{m_i} \in M_{\mathbf{v}} \otimes \mathbb{Z}/p\mathbb{Z}$ is non-zero for $i = 1, 2$. Therefore the intersection $\mathbf{b}_{v_1} \bigcap \mathbf{b}_{v_2}$ is non-empty if and only if $f_n(v_1, v_2) = 0$. The intersection $\mathbf{b}_{v_1} \bigcap \mathbf{b}_{v_2}$ then equals the building of a subgroup $SO(2r-1, \mathbb{Q}_p) \subset SO(2r+1, \mathbb{Q}_p)$. All non-empty intersections of buildings $\mathbf{b}_{v_i} \in \mathcal{T}_r$ are therefore transversal. It follows that the set $\mathcal{T}_r$ is an almost complete transversal system if and only if the union $|\mathcal{T}_r| = \bigcup_{\mathbf{b} \in \mathcal{T}_r} \mathbf{b}$ contains all vertices $\mathbf{v} \in \mathcal{B}_G$ that are of type $\tau(\mathbf{v}) = 0$. This is the case for $r = 1, 2, 3$.

Let us now discuss the boundary $|B_1(\mathcal{T})|$ of the transversal sytem $\mathcal{T}_r$ for $r = 1, 2, 3$. Let $\mathbf{b}_v, \mathbf{b}_{v'} \in \mathcal{T}_r$ be subbuildings such that the intersection $\mathbf{b}_v \bigcap \mathbf{b}_{v'}$ is empty and the distance $d_{\mathcal{B}_G}(\mathbf{b}, \mathbf{b}')$ is minimal for pairs of non-intersecting buildings in $\mathcal{T}_r$. The minimality of the distance between the buildings $\mathbf{b}_v$ and $\mathbf{b}_{v'}$ is equivalent to the existence of special vertices $\mathbf{v} \in \mathbf{b}_v$ and $\mathbf{v}' \in \mathbf{b}_{v'}$ (of different type, i.e. $\tau(\mathbf{v}) \neq \tau(\mathbf{v}')$) such that there exists an edge $\mathbf{e}$ in the building $\mathcal{B}_G$ containing the vertices $\mathbf{v}$ and $\mathbf{v}'$. This is equivalent to the existence of a special vertex $\mathbf{v} \in \mathbf{b}_v$ such that after replacing v' by the lowest multiple $p^m v'$, $m \in \mathbb{Z}$, such that $v' \in \Lambda_{\mathbf{v}} \subset M_{\mathbf{v}}$ the equality $f_{2r+1}(v', v') = p^2$ holds. Moreover, the prime p does not divide $f_{2r+1}(v, v')$, otherwise the vectors v and v'/p would be contained in some common $\mathbb{Z}$-lattice and the buildings $\mathbf{b}_v$ and $\mathbf{b}_{v'}$ would intersect.

The lattice $\Lambda_{\mathbf{v}}$ is isomorphic to the standard $\mathbb{Z}$-lattice $\Lambda \cong \mathbb{Z}^{2r+1}$. Let f_i, $i = 1, \ldots, 2r+1$ be an orthonormal basis of $\Lambda_{\mathbf{v}}$. We may assume that $v = f_1$. Then $v' = \sum_{i=1}^{2r+1} m_i f_i$ with $\sum_{i=1}^{2r+1} m_i^2 = p^2$. The stabiliser in H_v of the vector v' equals the stabiliser of the vector $v'' := v' - m_1 f_1 \in \Lambda_{\mathbf{v}}$. Since $p \not| f_{2r+1}(v, v') = m_1$, the prime p does not divide $f_{2r+1}(v'', v'') = p^2 - m_1^2$. If $p^2 - m_1^2$ is a square, then v'' is a multiple of one of the vectors f_i. In any case the stabiliser in $H_v(\mathbb{Q}_p)$ of the vector v'' is a subgroup $SO(2r-1, \mathbb{Q}_p)$ of $H_v(\mathbb{Q}_p) \cong SO(2r, \mathbb{Q}_p)$. Therefore $\{z \in \mathcal{B}_G \mid d_{\mathcal{B}_G}(z, \mathbf{b}_v) = d_{\mathcal{B}_G}(z, \mathbf{b}_{v'}) = d_{\mathcal{B}_G}(\mathbf{b}_v, \mathbf{b}_{v'})/2\}$ defines a subbuilding in $\mathcal{B}_G$ that corresponds to the group $H_v(\mathbb{Q}_p) \bigcap H_{v'}(\mathbb{Q}_p) \cong SO(2r-1, \mathbb{Q}_p)$. This shows that the simplicial structure of the building $\mathcal{B}_G$ can be refined in such a way the boundary $|B_1(\mathcal{T}_r)| \subset \mathcal{B}_G$ is the union of a set of codimension one subbuildings belonging to subgroups $SO(2r-1, \mathbb{Q}_p) \subset SO(2r+1, \mathbb{Q}_p)$.

Let us conclude this section by describing in somewhat more detail these transversal systems $\mathcal{T}_r$ for $r = 1, 2, 3$. For a special vertex $\mathbf{v}$ the $\mathbb{Z}$-lattice $\Lambda_{\mathbf{v}}$ is isomorphic to $\mathbb{Z}^{2r+1}$. Therefore a special vertex $\mathbf{v}$ is contained in exactly $2r+1$ subbuildings (or apartments if $r = 1$) $\mathbf{b}_v \subset \mathcal{B}_G$ that are contained in $\mathcal{T}_r$. Since the group $\Gamma = SO(2r+1, \mathbb{Z}[1/p])$ acts transitively on the vectors $v \in \Lambda[1/p]$ of norm one, the group Γ acts transitvely on the subbuildings $\mathbf{b}_v$ that are contained in the transversal system $\mathcal{T}_r$.

In particular, if $r = 1$ all vertices are special and $|\mathcal{T}_1|$ contains all vertices of the tree $\mathcal{B}_G$. Since each apartment $A_v \ni \mathbf{v}$ contains two edges $\mathbf{e} \ni \mathbf{v}$, exactly six edges $\mathbf{e} \ni \mathbf{v}$ are contained in $|\mathcal{T}_1|$. If $p = 5$ then a vertex $\mathbf{v} \in \mathcal{B}_G$ is contained in six edges. In this case $\mathcal{B}_G = \bigcup_{A \in \mathcal{T}_1} A$ and the transversal system is actually complete. If the prime $p > 5$, then a vertex $\mathbf{v} \in \mathcal{B}_G$ is contained in $p + 1 > 6$ edges. Therefore $|\mathcal{T}_1|$ does not contain all edges $\mathbf{e} \in \mathcal{B}_G$, but $|\mathcal{T}_1|$ still contains all vertices of the tree $\mathcal{B}_G$.

11.9. Fundamental weights and stable points. Let λ_G be a fundamental weight for the group G and let V_{λ_G} be the corresponding G-module (possibly not defined over K, but over the splitting field L of G). Let $X_{\lambda_G} \subset \mathbb{P}(V_{\lambda_G})$ be the homogeneous variety $X_{\lambda_G} := G/P_{\lambda_G}$, where $P_{\lambda_G} \subset G$ is the maximal parabolic subgroup that corresponds to the weight λ_G.

The restriction of the weight λ_G to Φ_H is a weight λ_H. If the group H is not almost simple and $H = H_1 \times \ldots \times H_s$ with the groups H_i, $i = 1, \ldots, s$ almost simple, then we define λ_H as $\lambda_H := \lambda_{H_1} \oplus \ldots \oplus \lambda_{H_s}$ with each λ_{H_i} a weight for H_i. We choose the weight λ_G in such a way that the weight λ_{H_i} is a minuscule weight (i.e. a minimal fundamental weight) for each irreducible component Φ_{H_i} of Φ_H. By abuse of the terminology, we will also call the weight $\lambda_H = \lambda_{H_1} \oplus \ldots \oplus \lambda_{H_s}$ minuscule in this situation. The irreducible simply laced root sytems except E_8 all posses minimal fundamental weights

λ_H. Since λ_H is a minuscule weight, the only weights that occur in the H-module V_{λ_H} are the weights $w(\lambda_H)$ with $w \in W_H$. Here W_H denotes the Weyl group of the root system Φ_H. If the root system Φ_H is not irreducible, then $V_{\lambda_H} := V_{\lambda_{H_1}} \oplus \ldots \oplus V_{\lambda_{H_s}}$. The G-module V_{λ_G} as a H-module equals $V_{\lambda_G} = V_0 \oplus (V_{\lambda_H})^{n_{\lambda_H,\lambda_G}}$ for some integer $n_{\lambda_H,\lambda_G} > 0$. Here V_0 is an H-module on which the group H acts trivially.

Let us fix a submodule $V_{\lambda_H} \subset V_{\lambda_G}$. Let $\phi_0^{\lambda_G}$ and $\phi_{\lambda_H}^{\lambda_G}$ denote the orthogonal projections $\phi_0^{\lambda_G} : V_{\lambda_G} \to V_0$ and $\phi_{\lambda_H}^{\lambda_G} : V_{\lambda_G} \to V_{\lambda_H}$. By $\varphi_{\lambda_H}^{\lambda_G}$ we denote the map $\varphi_{\lambda_H}^{\lambda_G} : X_{\lambda_G} \to X_{\lambda_H} := X_{\lambda_{H_1}} \times \cdots \times X_{\lambda_{H_s}} \subset \mathbb{P}(V_{\lambda_{H_1}}) \times \cdots \times \mathbb{P}(V_{\lambda_{H_s}})$. If $n_{\lambda_H,\lambda_G} > 1$, then the maps $\phi_{\lambda_H}^{\lambda_G}$ and $\varphi_{\lambda_H}^{\lambda_G}$ depend on the choice of the submodule $V_{\lambda_H} \subset V_{\lambda_G}$.

Let $Y_{\lambda_G}^s$ denote the analytical variety $Y_{\lambda_G}^s \subset X_{\lambda_G}$ that consists of the points $x \in X_{\lambda_G}$ that are stable for all maximal K-split tori of G and let $Y_{\lambda_H}^s \subset X_{\lambda_H}$ be defined analogously.

If the complement of $Y_{\lambda_G}^s \subset X_{\lambda_G}$ is of codimension one, then there probably exist many Γ-invariant functions on $Y_{\lambda_G}^s$. If the codimension of the complement is larger than one, then every Γ-invariant function can be extended to all of X_{λ_G}. Since such a function is $G(K)$-invariant, it must be constant. Therefore the only Γ-invariant functions on $Y_{\lambda_G}^s$ are the constants in this case.

Examples where $Y_{\lambda_H}^s \subset X_{\lambda_H}$ has a complement of codimension one are Drinfeld's symmetric space $Y_{\lambda_H}^s = \Omega_r \subset \mathbb{P}^r$ for the group $H(K) = SL(r+1, K)$ of type A_r and the case where $X_{\lambda_H} \subset \mathbb{P}^{2r-1}$ is given by a quadratic equation in $2r$ variables for the groups H of type D_r.

We say that a rigid analytic variety $\mathcal{Z}$ can be *locally embedded* into a rigid analytic variety $\mathcal{X}$, if it has an admissable covering $\mathcal{C}$ by open admissable subspaces such that each element of $\mathcal{C}$ can be embedded into the variety $\mathcal{X}$.

11.10. Arithmetic groups and transversal systems. Let G be a quasisplit group defined over the field K. Let $\Phi_H \subset \Phi_G$ be the subset of longest roots. Let $H(K) \subset G(K)$ be the group generated by the additive groups U_α, $\alpha \in \Phi_H$. The group $H(K)$ has the same K-rank as $G(K)$. Then $H = G$ holds if and only if G is a K-split group with a Dynkin diagram that is simply laced (i.e. Φ_G is of type A, D, E). In that case the transversal system $\mathcal{T} = \{\mathcal{B}_G\}$ is trivial.

For a discrete co-compact arithmetic group $\Gamma \subset G(K)$ we expect exactly one of the following two statements to hold:

i) G(K) is a K-split group such that all roots of Φ_G have the same length. So Φ_G is of type A_r, D_r, E_6, E_7 or E_8.

ii) There exists a Γ'-invariant non-trivial almost complete transversal system $\mathcal{T}_{\Gamma'}$ of $H(K)$-buildings $\mathbf{b}_H$ with H as above for some subgroup $\Gamma' \subset \Gamma$ of finite index.

Let us now describe how one might construct such a Γ-invariant almost complete transversal system. Let λ_G be a fundamental weight for the group G such that the restriction of the weight λ_G to Φ_H is a minuscule weight λ_H. Since λ_H is a minuscule weight, the G-module V_{λ_G} as a H-module equals $V_{\lambda_G} = V_0 \oplus (V_{\lambda_H})^{n_{\lambda_H,\lambda_G}}$. We only consider groups such that there exists a fundamental weight λ_G with $n_{\lambda_H,\lambda_G} = 1$.

Let the arithmetic group Γ be defined by a positive definite form f on V_{λ_G}. The form f is defined over the ring of integers $\mathcal{O}_{\mathcal{K}}$. Here $\mathcal{K} \supset \mathbb{Q}$ is a finite Galois extension. For each vertex $\mathbf{v} \in \mathcal{B}_G$ one has an $\mathcal{O}_{\mathcal{K}}$-lattice $\Lambda_{\mathbf{v}}$ w.r.t. this form and Γ permutes the lattices $\Lambda_{\mathbf{v}}$ for $\mathbf{v} \in \mathcal{B}_G$.

Let the group $H' \subset G$, $H' \cong H$ be such that $\mathbf{v} \in \mathbf{b}_{H'}$. Then $V_{\lambda_G} = V_0 \oplus V_{\lambda_{H'}}$. If the root system Φ_H is not irreducible, then $H' = H'_1 \times \cdots \times H'_s$ with the groups H'_i almost simple for $i = 1, \ldots, s$. Then $\lambda_{H'} = \lambda_{H'_1} \oplus \cdots \oplus \lambda_{H'_s}$ and $V_{\lambda_{H'}} = \oplus_{1=1}^{s} V_{\lambda_{H'_i}}$. Let $\Lambda_{\mathbf{v},H'} \subset \Lambda_{\mathbf{v}}$ denote the sublattice $\Lambda_{\mathbf{v},H'} :=< \Lambda_{\mathbf{v}} \bigcap V_0 > \oplus < \Lambda_{\mathbf{v}} \bigcap V_{\lambda_{H'_1}} > \oplus \cdots \oplus < \Lambda_{\mathbf{v}} \bigcap V_{\lambda_{H'_s}} >$. Let $\imath_{\mathbf{v},H'} := [\Lambda_{\mathbf{v}} : \Lambda_{\mathbf{v},H'}]$ be the index and let $\imath_{\mathbf{v}} := min\{\imath_{\mathbf{v},H'} \mid \mathbf{v} \in \mathbf{b}_{H'}\}$ be the minimum index. Let $\mathcal{T}_\Gamma := \{\mathbf{b}_{H'} \mid \exists (\mathbf{v} \in \mathcal{B}_G, \tau(\mathbf{v}) = 0)\ \mathbf{v} \in \mathbf{b}_{H'}, \imath_{\mathbf{v},H'} = \imath_{\mathbf{v}}\}$. The set $\mathcal{T}_\Gamma$ seems to be the most likely candidate for being a Γ-equivariant almost complete transversal system.

The set $\mathcal{T}_\Gamma$ is entirely determined by the $\mathcal{O}_{\mathcal{K}}$-lattices $\Lambda_{\mathbf{v}}$ and $\Lambda_{\mathbf{v},H}$. In some sense it depends only on the number field $\mathcal{K}$ and the definition of the group G over the number field $\mathcal{K}$. Let K and K' be different completions of the number field $\mathcal{K}$, such that the groups $G(K)$ and $G(K')$ are of the same type. Let Γ and Γ' be the corresponding arithmetic discrete co-compact subgroups of $G(K)$ and $G(K')$, respectively. The set of Γ-orbits $\mathcal{T}_\Gamma/\Gamma$ does not depend on the particular local field K that is the completion of the number field $\mathcal{K}$. In particular, the sets $\mathcal{T}_\Gamma/\Gamma$ and $\mathcal{T}_{\Gamma'}/\Gamma'$ are isomorphic.

Of course, we have in no way proved either the existence nor the uniqueness of a Γ-invariant almost complete transversal system.

11.11. Deligne-Lusztig varieties and étale coverings. Let $\Gamma \subset G(K)$ be an arithmetic discrete co-compact subgroup and let $\mathcal{T}_\Gamma$ be a Γ-invariant almost complete transversal system of $H(K)$-subbuildings $\mathbf{b}_H \subset \mathcal{B}_G$. Our aim is to construct a rigid analytic variety $\mathcal{Y}_\Gamma$ on which the group Γ acts discretely such that the quotient $\mathcal{Y}_\Gamma/\Gamma$ is a proper variety that is algebraizable. Such a construction depends very much on the transversal system $\mathcal{T}_\Gamma$.

Locally the space $\mathcal{Y}_\Gamma$ is an étale covering of a rigid analytic variety $\mathcal{Y}_H$. The group $H(K)$ acts on the analytic variety $\mathcal{Y}_H$ and the quotients $\mathcal{Y}_H/\Gamma_H$ are proper algebraizable varieties for discrete co-compact subgroups $\Gamma_H \subset H(K)$. The only known example of such a space $\mathcal{Y}_H$ is Drinfel'ds symmetric space for the groups H of type A.

We assume that both $\mathcal{Y}_\Gamma$ and $\mathcal{Y}_H$ have a reduction that consists of proper components corresponding to the vertices $\mathbf{v}$ in the buildings $\mathcal{B}_G$ and $\mathbf{b}_H$, respectively. We compare suitably choosen open subvarieties of the components of the reduction of $\mathcal{Y}_\Gamma$ and $\mathcal{Y}_H$ at a special vertex $\mathbf{v} \in \mathbf{b}_H \subset \mathcal{B}_G$. This gives us some relation between a $H(k)$-invariant variety Y_H and a $G(k)$-invariant variety Y_G, that are satisfied when such a analytic variety $\mathcal{Y}_\Gamma$ exists. Therefore these conditions are necessary for the existence of an analytic variety $\mathcal{Y}_\Gamma$.

We will assume that the varieties can be embedded into some projective homogeneous variety. Let λ_G be a weight of Φ_G, such that the weight $\lambda_H := \lambda_G|_{\Phi_H}$ is minuscule. We again assume that $n_{\lambda_H,\lambda_G} = 1$ holds.

Let us state these conditions:

F(i) There exists a $H(k)$-invariant variety Y_H that can be embedded in $Y^s_{\lambda_H} \otimes k \subset X_{\lambda_H} \otimes k$.

F(ii) There exists a $H(k)$-equivariant étale covering Z_H of Y_H that can be embedded in $X_{\lambda_G} \otimes k$. The restriction of the $H(k)$-equivariant map $\varphi^{\lambda_G}_{\lambda_H}$ to Z_H gives the étale map $Z_H \to Y_H$.

F(iii) Let $Y_G := \bigcap_{g \in G(k)} g(Z_H)$. Then Y_G is a non-empty open subvariety of Z_H. The variety Y_G can be embedded in $Y^s_{\lambda_G} \otimes k$.

For the group $SU(3, L)$ both the varieties Y_H and Y_G are Deligne-Lusztig varieties. It seems reasonable to expect that this holds more generally.

Let us therefore recall some definitions concerning Deligne-Lusztig varieties. Let $X(w)_G := \{g \in G/B \mid g^{-1}F(g) \in BwB\}$ denote the Deligne-Lusztig variety for G belonging to the element $w \in W$. Let $Y(w)_G := \{gU \in G/U \mid g^{-1}F(g) \in UwU\}$. We have a finite $G(k)$-equivariant étale map $Y(w)_G \to X(w)_G$.

Let us assume that in condition F(i) above, the variety Y_H is a Deligne-Lusztig variety $X(w)_H$. Let $Z(w)_H$ a suitably chosen algebraic variety, such that $Y(w)_H \to Z(w)_H \to X(w)_H$, where the arrows denote finite $H(k)$-equivariant étale maps. Then we take as Z_H a connected component $Z(w)_H^\circ$ of the variety $Z(w)_H$. The variety Y_G then is the variety $Y_G := \bigcap_{g \in G(k)} g(Z_H) = \bigcap_{g \in G(k)} g(Z(w)_H^\circ)$.

If the root system Φ_H is of type A, then the variety Y_H comes from Drinfel'ds symmetric space Ω_H and equals $\Omega_H \otimes k$. The variety Y_H then is the Deligne-Lusztig variety $X(w)_H$, where the element w is a Coxeter element of the Weyl group of $H(k)$ (See [OR]).

An element $w \in W$ is called a *Coxeter element* if it is the product of the r reflections that correspond to the simple roots of a basis of the root system Φ. Let α_i, $i = 1, \dots, r$ be a simple basis of the root sytem Φ. Let $w_i \in W$ be the reflection belonging to the simple root α_i, then $w = w_1 \cdots w_r$ is a Coxeter element. All Coxeter elements are conjugated in the Weyl group W. The variety Y_G in statement F(iii) above has then the same dimension r as $X(w)_H$ and is irreducible. Therefore it is possible that $Y_G = X(w')_G$ holds, where the element w' is a Coxeter element of the Weyl group of $G(k)$.

11.12. Example: Curves. Let us describe the varieties Y_H, Z_H and Y_G for the groups $G(k) = SU(3, \ell)$ and $H(k) = SU(2, \ell)$. Let x_0, x_1, x_2 be coordinates of $\mathbb{P}^2_\ell$ and let $\mathbb{P}^1_\ell \subset \mathbb{P}^2_\ell$ be given by $x_0 = 0$. The group $SU(3, \ell)$ acts on $\mathbb{P}^2_\ell$ preserving the hermitian form $h_3(x, y) = x_1 y_2^q + x_2 y_1^q + x_0 y_0^q$. The subgroup $SU(2, \ell) \subset SU(3, \ell)$ fixes the point $(x_0, 0, 0)$ and acts on the $\mathbb{P}^1_\ell \subset \mathbb{P}^2_\ell$ defined by $x_0 = 0$ preserving the hermitian form. The varieties X_H, Z_H and Y_G are as follows:

$$Y_H := \{x \in \mathbb{P}^2_\ell \mid x_0 = 0\} - \{x \in \mathbb{P}^2(\ell) \mid h_3(x, x) = 0\} \cong (\mathbb{P}^1_k - \mathbb{P}^1(k)) \otimes \ell.$$

$$Z_H := \{x \in \mathbb{P}^2_\ell \mid h_3(x, x) = 0\} - \{x \in \mathbb{P}^2(\ell) \mid x_0 = 0, h_3(x, x) = 0\}.$$

$$Y_G := \{x \in \mathbb{P}^2_\ell \mid h_3(x, x) = 0\} - \{x \in \mathbb{P}^2(\ell) \mid h_3(x, x) = 0\}.$$

Another interesting example is the case where $G(k) = SO(3, k)$ and the group $H(k) = T(k)$ is a maximal k-split torus of $G(k)$. The group $G(k)$ acts on $\mathbb{P}^2_k$ with coordinates x_0, x_1, x_2 preserving the quadratic form f defined by $x_0^2 + x_1 x_2 = 0$. The varieties X_H, Z_H and Y_G for this case are as follows:

$$Y_H := \{x \in \mathbb{P}^2_k \mid x_0 = 0,\ x_1 x_2 \neq 0\} \cong k^*.$$

$$Z_H := \{x \in \mathbb{P}^2_k \mid f(x, x) = 0\} - \{(0, x_1, 0), (0, 0, x_2)\}.$$

$$Y_G := \{x \in \mathbb{P}^2_k \mid f(x,x) = 0\} - \{x \in \mathbb{P}^2(k) \mid f(x,x) = 0\} \cong \mathbb{P}^1_k - \mathbb{P}^1(k).$$

11.13. Example: Split symplectic groups. Let G be the split symplectic group $Sp(2r,k)$. Then G is a k-split group of type $\Phi_G = C_r$. The subgroup $H(k) \subset G(k)$ equals the group $H(k) \cong SL(2,k)^r$.

For the group $Sp(4,k)$ we determine varieties Y_H, Z_H and Y_G such that conditions F(i)-F(iii) hold. Our construction also gives varieties for $r > 2$, however then our candidate for the variety Z_H fails to be an étale covering of the variety Y_H. Hence condition F(ii) does not hold for these varieties, if $r > 2$.

Let V denote the vector space $V \cong k^{2r}$. Let us choose a basis e_i, e_{-i}, $i = 1, \dots, r$ of the vector space V and corresponding coordinates x_i, x_{-i}, $i = 1, \dots, r$. We define a symplectic form h_s on the vector space V by $h_s(x,y) := \sum_{i=1}^r (x_i y_{-i} - x_{-i} y_i)$. The group $Sp(2r,k)$ acts on the projective space $\mathbb{P}(V) \cong \mathbb{P}^{2r-1}_k$ and preserves the symplectic form h_s.

We define an $H(k)$-equivariant map $\varphi_H : \mathbb{P}^{2r-1}_k \longrightarrow (\mathbb{P}^1_k)^r$, by mapping x to the point $(x_1, x_{-1}) \times \cdots \times (x_r, x_{-r}) \in (\mathbb{P}^1_k)^r$.

Our Deligne-Lusztig variety for the group $G(k)$ in $\mathbb{P}^{2r-1}_k$ consists of the points $x \in \mathbb{P}^{2r-1}_k$ such that $< x^{q^j} \mid j = 0, \dots, r >$ is a maximal isotropic subspace that is not k-rational for the symplectic form. They are contained in the variety X_s defined by the equations $h_s(x, x^{q^j}) = \sum_{i=1}^r (x_i x_{-i}^{q^j} - x_{-i} x_i^{q^j}) = 0$, $j = 1, \dots, r-1$. The open subset defined by the inequality $h_s(x, x^{q^r}) = \sum_{i=1}^r (x_i x_{-i}^{q^r} - x_{-i} x_i^{q^r}) \neq 0$ equals our Deligne-Lusztig variety.

We denote by $\varphi_{s,H}$ the restriction of φ_H to X_s. Let $X_{s,H} := \varphi_{s,H}^{-1}(\Omega_1^r) \subset X_s$, where $\Omega_1^r := (\mathbb{P}^1_k)^r - (\mathbb{P}^1(k))^r$. Let $X_s^\circ := \bigcap_{g \in G(k)} g(X_{s,H})$.

Let us show that $X_s^\circ \subset X_s$ is the Deligne-Lusztig variety defined above for the group $G(k)$. For a point $x \in X_s$ we denote by $V(x) \subset V$ the plane $V(x) :=< x_i e_i + x_{-i} e_{-i} \mid i = 1, \dots, r >$. If $\varphi_{s,H}(x) \in (\mathbb{P}^1(k))^r = (\mathbb{P}^1_k)^r - \Omega_1^r$, then the vectors $x_i e_i + x_{-i} e_{-i} \in V$ are defined over k for $i = 1, \dots, r$. Therefore the plane $V(x) \subset V$ is isotropic of rank $\leq r$ and defined over the field k. On the other hand, if the point x is contained in a k-rational isotropic plane of rank $\leq r$, then there exists k-rational (isotropic) vectors f_i, $i = 1, \dots, r$, such that $x = \sum_{i=1}^r x_i f_i$. Extending the set f_i by (isotropic) vectors f_{-i}, $i = 1, \dots, r$ defined over k such that $h_s(f_i, f_{-i}) = 1$ and $h_s(f_j, f_{-i}) = h_s(f_i, f_{-j}) = 0$ for $i \neq j$ defines a group $H'(k) \cong H(k)$ preserving the subspaces $< f_i, f_{-i} >$. For this group $H'(k)$ the point $\varphi_{s,H'}(x)$ is contained in $(\mathbb{P}^1(k))^r = (\mathbb{P}^1_k)^r - \Omega_1^r$. Hence $x \notin X_s^\circ$.

The map $\varphi_{s,H} : X_{s,H} \to \Omega_1^r$ is étale of degree $q+1$ for $r = 2$. For

$r > 2$ this map is no longer étale. Therefore we have established that for the group $Sp(4,k)$ the varieties $Y_H = \Omega_1^2$, $Z_H = X_{s,H}$ and $Y_G = X_s^\circ$ satisfy the conditions F(i)-F(iii).

For $r = 2$ the variety X_s° is actually the Deligne-Lusztig variety belonging to a Coxeter element $w \in W$ for the group $Sp(4,k)$ (See [Ro] prop. 7.4). It seems likely that this remains true for values $r > 2$.

11.14. Example: Split unitary groups. Let G be the unitary group $SU(2r,\ell)$ with $\ell \supset k$ a quadratic extension. Then G is a quasisplit group defined over k with root system Φ_G of type C_r. The group $H(k) \subset G(k)$ equals $H(k) = SU(2,\ell)^r \cong SL(2,k)^r$.

For the group $SU(4,\ell)$ we determine varieties Y_H, Z_H and Y_G such that conditions F(i)-F(iii) hold. Our construction also gives varieties for $r > 2$, however then our candidate for the variety Z_H is not an étale covering of the variety Y_H. Hence condition F(ii) fails to hold for these varieties, if $r > 2$.

Let V_ℓ be the vector space $V_\ell := \ell^{2r}$. Let us choose a basis e_i, e_{-i}, $i = 1,\ldots,r$ of the vector space V_ℓ and corresponding coordinates x_i, x_{-i}, $i = 1,\ldots,r$. Let h_u be the unitary form on V_ℓ defined by $h_u(x,y) := \sum_{i=1}^r (x_i y_{-i}^q + x_{-i} y_i^q)$. The group group $SU(2r,\ell)$ acts on the projective space $\mathbb{P}(V_\ell) \cong \mathbb{P}_\ell^{2r-1}$ preserving the unitary form h_u.

We define an $H(k)$-equivariant map $\varphi_H : \mathbb{P}_\ell^{2r-1} \longrightarrow (\mathbb{P}_\ell^1)^r$, by mapping x to the point $(x_1, x_{-1}) \times \cdots \times (x_r, x_{-r}) \in (\mathbb{P}_\ell^1)^r$.

Our Deligne-Lusztig variety for the group $G(k)$ in $\mathbb{P}_\ell^{2r-1}$ consists of the points $x \in \mathbb{P}_\ell^{2r-1}$ such that $< x^{q^j} \mid j = 0,\ldots,r-1 >$ is a maximal isotropic subspace that is not defined over the field ℓ for the unitary form.

They are contained in the variety X_u defined by the equations $h_u(x, x^{q^j}) = \sum_{i=1}^r (x_i x_{-i}^{q^{j+1}} + x_{-i} x_i^{q^{j+1}}) = 0$, $j = 0,\ldots,r-1$. The open subset defined by the inequality $h_u(x, x^{q^r}) = \sum_{i=1}^r (x_i x_{-i}^{q^{r+1}} + x_{-i} x_i^{q^{r+1}}) \neq 0$ gives the required variety.

We denote by $\varphi_{u,H}$ the restriction of φ_H to X_u. Let $X_{u,H} := \varphi_{u,H}^{-1}(\Omega_{1,\ell}^r) \subset X_u$. Here $\Omega_{1,\ell} \subset \mathbb{P}_\ell^1$ denotes $\mathbb{P}_\ell^1$ with the ℓ-valued isotropic points removed. Let $X_u^\circ := \bigcap_{g \in G(k)} g(X_{u,H})$. Then $X_u^\circ \subset X_u$ is the Deligne-Lusztig variety defined above for the group $G(k)$. This can be proved as in the case of the symplectic group above.

The map $\varphi_{u,H} : X_{u,H} \to \Omega_{1,\ell}^r$ is étale if $r = 2$. For $r > 2$ this map is no longer étale. Therefore for the group $SU(4,\ell)$ we have obtained varieties $Y_H = \Omega_{1,\ell}^2$, $Z_H = X_{u,H}$ and $Y_G = X_u^\circ$ that satisfy conditions F(i)-F(iii).

For $r = 2$ the variety X_u° is actually the Deligne-Lusztig variety belonging to a Coxeter element $w \in W$ for the group $SU(4, \ell)$ (See [Ro] prop. 6.6). It seems unlikely that this remains true for values $r > 2$, because one uses the Frobenius $F: x \to x^{q^2}$ to define this Deligne-Lusztig variety.

Everything above carries more or less over to the group $G(k) = SU(2r+1, \ell)$. This group has root system Φ_G of type BC_r. The root system Φ_H is of type A_1^r and $H(k) = SU(2, \ell)^r$. The group $G(k)$ acts on a vector space $V \cong \ell^{2r+1}$ preserving a unitary form. For $r = 1$ one obtains the example of the hermitian curve for the group $SU(3, \ell)$. For $r > 1$ the candidate for the variety Z_H is such that condition F(ii) does not hold.

11.15. Rigid analytic spaces and étale coverings. We expect the properties F(i)-F(iii) above concerning the $H(k)$-invariant varieties Y_H and Z_H and the $G(k)$-invariant variety Y_G over the finite field k, to result in analytic varieties that have analogous properties over the local field K, provided the root system Φ_H is of type A:

- L(i) There exists a rigid analytical variety $\mathcal{Y}_H$ that can be locally embedded into $Y^s_{\lambda_H}$ on which the group $H(K)$ acts. The components of the reduction of $\mathcal{Y}_H$ at special vertices $\mathbf{v} \in \mathbf{b}_H$ are completions of the variety Y_H. The quotients $\mathcal{Y}_H/\Gamma_H$ are proper and algebraizable for discrete co-compact subgroups $\Gamma_H \subset H(K)$.
- L(ii) There exists a finite $H(K)$-equivariant étale covering $\mathcal{Z}_H \to \mathcal{Y}_H$ with Galois group $Gal(Z_H/Y_H)$. The components of the reduction of $\mathcal{Z}_H$ at special vertices are completions of the variety Z_H and therefore completions of Y_G. The variety $\mathcal{Z}_H$ is such that it can be locally embedded into $Y^s_{\lambda_G}$.
- L(iii) There exists an open admissable open subspace $\mathcal{Z}_H^\circ \subset \mathcal{Z}_H$, such that the varieties $\mathcal{Z}_H^\circ$ for the groups $H(K)$ that correspond to the buildings $\mathbf{b}_H$ in the Γ-invariant transversal system $\mathcal{T}_\Gamma$ can be glued together into a rigid analytic variety $\mathcal{Y}_\Gamma^\circ$ on which the group Γ acts discretely.
- L(iv) The quotient $\mathcal{Y}_\Gamma^\circ/\Gamma$ can be compactified, such that the compactfication is a proper algebraizable variety.

The rigid analytic varieties $\mathcal{Y}_H$ and $\mathcal{Z}_H$ are analogs over the local field K of the algebraic varieties Y_H and Z_H over the residue field k. The open admissable subvariety $\mathcal{Z}_H^\circ \subset \mathcal{Z}_H$ is the analog of the subvariety $Y_G \subset Z_H$.

The condition that $Y_G = \bigcap_{g\in G(k)} g(Z_H)$ holds should be enough to ensure us that the intersection $\mathcal{Z}_H^\circ \bigcap \mathcal{Z}_{H'}^\circ$ is empty for subbuildings $\mathbf{b}_H, \mathbf{b}_{H'} \in \mathcal{T}_\Gamma$ that have an empty intersection. Moreover, if the intersection $\mathbf{b}_H \bigcap \mathbf{b}_{H'}$ is non-empty, then this condition should enable us to identify the parts of $\mathcal{Z}_H^\circ$ and $\mathcal{Z}_{H'}^\circ$ that correspond to the intersection $\mathbf{b}_H \bigcap \mathbf{b}_{H'}$. In general, the variety $\mathcal{Y}_\Gamma^\circ$ consists of infinitely many connected components. The connected components correspond to the connected components of the subset $|\mathcal{T}_\Gamma| \subset \mathcal{B}_G$.

Since the quotients $\mathcal{Y}_H/\Gamma_H$ are assumed to be algebraizable, there exist Γ_H-invariant functions on the variety $\mathcal{Y}_H$. These functions lift to Γ_H-invariant functions on $\mathcal{Z}_H$. The glueing of the subspaces $\mathcal{Z}_H^\circ$ for the groups $H(K)$ such that $\mathbf{b}_H \in \mathcal{T}_\Gamma$ should be such that the Γ_H-invariant functions glue together and give Γ-invariant functions on the union $\mathcal{Y}_\Gamma^\circ$. The variety $\mathcal{Y}_\Gamma^\circ$ locally looks like the étale covering $\mathcal{Z}_H$ of $\mathcal{Y}_H$. This should enable us to compactify the variety $\mathcal{Y}_\Gamma^\circ$ by adding suitable open admissable subspaces of the variety $\mathcal{Y}_H$ for the vertices $\mathbf{v} \in \mathcal{B}_G$ that are contained in the boundary $|B_1(\mathcal{T}_\Gamma)|$ of the transversal system.

Therefore we expect that conditions F(i)-F(iii) together with condition L(i) imply that the conditions L(ii)-L(iv) hold. Condition L(i) has to be established indepently for each group $H(K)$.

If the root system Φ_H is of type A, then the space $\mathcal{Y}_H$ is of course Drinfel'ds symmetric space Ω_H that belongs to the group $H(K)$. For the groups $H(K)$ with root system Φ_H of type D_r, $r > 3$, E_6, E_7 or E_8, the existence of a space $\mathcal{Y}_H$ is doubtful. Though a space $\mathcal{Y}_H$ with an $H(K)$-action that is based on a variety Y_H that is the Deligne-Lusztig variety $X(w)_H$ belonging to a Coxeter element w seems unlikely, there remains a small possibility that a useful space $\mathcal{Y}_H$ on which $H(K)$ acts, based on taking as the variety Y_H the étale covering $Y(w)_H$ of $X(w)_H$, might exist for groups $H(K)$ of type D and E.

11.16. Example: Étale coverings of Ω_r. Let us briefly discuss a finite étale covering of Drinfeld's symmetric space $\Omega_r := \mathbb{P}_K^r - \{K-\text{rational hyperplanes}\}$. Let $L' \supset K$ be the unramified extension of degree $r+1$. Let σ be a generator of the Galois group $Gal(L'/K)$. We let σ act on the coordinates of $\mathbb{P}_K^r(L')$ as usual.

A hyperplane in $\mathbb{P}_K^r(L')$ is K-rational if and only if $det(x, \sigma(x), \ldots, \sigma^r(x)) = 0$ for all L'-valued points x in the hyperplane. In the reduction the Galois element σ acts on the coordinates $\overline{x_i}$ by $\overline{\sigma(x_i)} = \overline{x_i^q}$ for $i = 1, \ldots, r+1$. Therefore $\overline{det(x, \sigma(x), \ldots, \sigma^r(x))}$ is a polynomial of degree $(q^{r+1}-1)/(q-1)$.

It is equal to the product of the linear equations that define the k-rational hyperplanes in $\mathbb{P}_k^r$. The polynomial evaluated at a point $\overline{x}$ is unequal to zero, if the point $\overline{x}$ is contained in the Deligne-Lusztig variety $X(w)_{SL_{r+1}(k)}$ for a Coxeter element w in the Weyl group.

The space Ω_r has a reduction consisting of components that are compactifications of $X(w)_{SL_{r+1}(k)}$. This is the standard reduction consisting of components isomorphic to $\mathbb{P}_k^r$ with all k-linear subspaces blown up. The space $Y(w)_{SL_{r+1}(k)}$ can in this case be defined by $\{x \in \mathbb{A}_k^{r+1} \mid (det(x, x^q, \ldots, x^{q^r})^{q-1} = (-1)^r\}$. It consists of $q-1$ connected components. We expect that there exists an $SL_{r+1}(K)$-equivariant étale covering of Ω_r that has a reduction consisting of components that are isomorphic to compactifications of a connected component $Y(w)^\circ_{SL_{r+1}(k)}$ of the space $Y(w)_{SL_{r+1}(k)}$. (I suspect that this is known to the experts.)

The group $SL_{r+1}(K)$ is contained in a quasisplit group $G(K)$ as the group $H(K)$ corresponding to the long roots if and only if $r = 1, 2, 3$. One has $SL_2(K) \cong SU(2, L) \subset SU(3, L)$, $SL_3(K) \subset G_2 \subset {}^3D_4$ and $SL_4(K) \cong SO(6) \subset SO(7)$. The group ${}^3D_4$ has a root system of type G_2. We expect that the varieties $X(w'')_{^3D_4} \otimes \ell$, $X(w')_{G_2} \otimes \ell$ and $Y(w)^\circ_{SL_3(k)} \otimes \ell$ are related for suitable elements w'' and w' in the Weyl groups of the groups of type ${}^3D_4$ and G_2, respectively. Here $\ell \supset k$ denotes a suitable finite extension (most likely of degree three) of the field k. It seems reasonable to expect that Ω_2 has an étale covering whose reduction consists of components related to (maybe isomorphic to) a compactification of the Deligne-Lusztig variety $X(w'')_{^3D_4}$. The fact that the groups of type G_2 preserve an alternating trilinear form and that the space $Y(w)^\circ_{SL_3(k)}$ is defined by using a sesquilinear trilinear form based on $det(x, x^q, x^{q^2})$ with $x \in \mathbb{A}_k^3$, gives some support to this idea.

11.17. Open subsets of the affine building. We define a covering $\mathcal{C}_{\mathcal{B}_G}$ of the building $\mathcal{B}_G$ by open convex subsets. The open subsets correspond to the simplices of the building $\mathcal{B}_G$. These open subsets can be used to define an open admissable covering of a rigid analytic variety on which a discrete group $\Gamma \subset G(K)$ acts.

Let $\mathbf{v} \in \mathcal{B}_G$ be a vertex. Then we denote by $\parallel \mathbf{v} \parallel_{\mathcal{B}_G} \subset \mathcal{B}_G$ the open subset $\parallel \mathbf{v} \parallel_{\mathcal{B}_G} := \mathcal{B}_G - \{\tau \in \mathcal{B}_G \text{ a simplex} \mid \mathbf{v} \notin \tau\}$. To any simplex $\sigma \in \mathcal{B}_G$ we associate the open subset $\parallel \sigma \parallel_{\mathcal{B}_G} := \bigcap_{\mathbf{v} \in \sigma} \parallel \mathbf{v} \parallel_{\mathcal{B}_G}$. The open subsets $\parallel \sigma \parallel_{\mathcal{B}_G} \subset \mathcal{B}_G$ are convex.

This defines a covering $\mathcal{C}_{B_G} := \{\parallel \sigma \parallel_{\mathcal{B}_G} \mid \sigma \in \mathcal{B}_G \text{ a simplex}\}$ by open convex subsets. The open subsets of the building have the property that the intersection $\parallel \mathbf{v} \parallel_{\mathcal{B}_G} \bigcap \parallel \mathbf{v}' \parallel_{\mathcal{B}_G}$ is non-empty if and only if the vertices $\mathbf{v}$ and $\mathbf{v}'$ form a simplex in the building. Moreover, $\parallel \sigma \parallel_{\mathcal{B}_G} \subset \parallel \tau \parallel_{\mathcal{B}_G}$ holds if and only if $\tau \subset \sigma$ for simplices $\sigma, \tau \subset \mathcal{B}_G$.

For an almost complete transversal system $\mathcal{T}$ of $H(K)$-subbuildings the subset $\parallel \mathcal{T} \parallel_{\mathcal{B}_G} := \bigcup_{\mathbf{v} \in |\mathcal{T}|} \parallel \mathbf{v} \parallel_{\mathcal{B}_G}$ is open in the building $\mathcal{B}_G$ with complement the boundary $|B_1(\mathcal{T})|$ of the transversal sytem $\mathcal{T}$.

These same definitions by open subsets that correspond to simplices are still applicable, when one refines the simplicial structure of the building $\mathcal{B}_G$ somewhat by subdividing some of the simplices that are not contained in $|\mathcal{T}| \subset \mathcal{B}_G$. Such subdivisions are sometimes needed to obtain a boundary $|B_1(\mathcal{T})|$ with nice properties.

Let $G(K) = H(K)$ be a group such that the root system $\Phi_G = \Phi_H$ is of type A. The group $H(K)$ acts on Drinfel'ds symmetric space Ω_H. One has Drinfel'ds reduction map $\psi_{\mathbf{b}} : \Omega_H \to \mathbf{b}_H = \mathcal{B}_H$. For a simplex $\sigma \in \mathbf{b}$ one defines the subset $\Omega_{\sigma,H} := \{x \in \Omega_H \mid \psi_{\mathbf{b}}(x) \in \parallel \sigma \parallel_{\mathbf{b}}\}$ of Ω_H. This gives an open admissable covering of the symmetric space Ω_H. Similarly, one can construct open admissable coverings of any finite étale $H(K)$-equivariant covering $\phi : \mathcal{Z}_H \to \Omega_H$ by using the reduction map $\psi_{\mathbf{b}} \circ \phi : \mathcal{Z}_H \to \mathbf{b}_H$.

11.18. Example: Mumford curves. The construction described above can be used to obtain the well-known uniformisation of Mumford curves. The example is trivial, but instructive. Starting with the uniformisation of Tate curves by K^*, one obtains the uniformisation of Mumford curves by $\Omega_{1,K} := \mathbb{P}^1_K - \mathbb{P}^1(K)$.

Let $G(K)$ be the group $SO(3, K)$. Let $\Gamma \subset SO(3, K)$ be a discrete co-compact subgroup and let $\mathcal{T}_\Gamma$ be a Γ-invariant almost complete transversal system of apartments $A \subset \mathcal{B}_G$. The building $\mathcal{B}_G$ in this case is a tree. We assume that $|\mathcal{T}_\Gamma|$ contains all vertices $\mathbf{v} \in \mathcal{B}_G$. This is a necessary requirement for the construction.

We first describe the refined simplicial structure of the tree $\mathcal{B}_G$. Since $\mathcal{T}_\Gamma$ contains all vertices $\mathbf{v} \in \mathcal{B}_G$, the boundary $B_1(\mathcal{T}_\Gamma)$ is empty, if one uses the standard simplicial structure of the tree $\mathcal{B}_G$. If the transversal system $\mathcal{T}_\Gamma$ is not complete, then some edges $\mathbf{e} \in \mathcal{B}_G$ are not contained in $|\mathcal{T}_\Gamma|$. Let $\mathbf{v}, \mathbf{v}'$ be the two vertices that are contained in the edge $\mathbf{e} \notin |\mathcal{T}_\Gamma|$. There exist apartments $A, A' \in \mathcal{T}_\Gamma$ such that $\mathbf{v} \in A$ and $\mathbf{v}' \in A'$. The intersection of the tori $T(K)$ and $T'(K)$ that correspond to the apartments A and A', respectively, is a trivial group $\{id.\}$. We have to add a vertex, i.e. the building of this trivial group, to the tree $\mathcal{B}_G$ in such a way that it separates the apartments A and A' and is equidistant to both apartments. Therefore we have to add the midpoint of the edge $\mathbf{e}$ as a vertex to the tree $\mathcal{B}_G$. It follows that the refined simplicial structure of the tree $\mathcal{B}_G$ is obtained by adding as vertices the midpoints of the edges $\mathbf{e} \notin |\mathcal{T}_\Gamma|$. The edges $\mathbf{e} \notin |\mathcal{T}_\Gamma|$ are therefore subdivided into two edges.

Let us now describe the rigid analytic variety $\mathcal{Y}_A := \Omega_A$ and the étale covering $\mathcal{Z}_A$ of Ω_A belonging to the torus $T(K)$ that acts on the apartment $A \subset \mathcal{T}_\Gamma$. The space $\Omega_A \cong K^*$ is obtained from the usual action of the torus on $\mathbb{P}^1_K$ by removing the two points that are fixed by the torus action. A discrete co-compact subgroup $\Gamma_T \subset T(K)$ gives a proper quotient Ω_A/Γ_T, which is a Tate curve. To construct the variety $\mathcal{Z}_A$, we let the group $SO(3,K)$ act on $\mathbb{P}^2_K$ preserving the quadratic equation $x_0^2 + x_1x_2 = 0$. Let the coordinates x_0, x_1, x_2 be chosen in such a way that the torus T acts diagonally on $\mathbb{P}^2_K$ w.r.t. these coordinates and such that the point $(x_0, 0, 0)$ is the unique point in $\mathbb{P}^2_K$ that is fixed by the action of the torus T. For the standard apartment $A \in \mathcal{T}_\Gamma$ that corresponds to the torus $T \in SO(3,K)$, we embed the space Ω_A into $\mathbb{P}^2_K$ as the space $\Omega_A := \{(0, x_1, x_2) \in \mathbb{P}^2_K \mid x_1, x_2 \neq 0\} \subset \mathbb{P}^2_K$. The space $\mathcal{Z}_A$ is defined as being $\mathcal{Z}_A := \{(x_0, x_1, x_2) \in \mathbb{P}^2_K \mid x_0^2 + x_1x_2 = 0,\ x_1, x_2 \neq 0\}$. Then one defines $\Omega_{gA} := g(\Omega_A)$ and $\mathcal{Z}_{gA} := g(\mathcal{Z}_A)$ for any apartment $gA \in \mathcal{T}_\Gamma$, where $g \in SO(3,K)$. The map $\varphi_A :\ \mathcal{Z}_A \to \Omega_A$ defined by $\varphi_A((x_0, x_1, x_2)) = (0, x_1, x_2)$ is a $T(K)$-equivariant étale map of degree two.

Let us now describe the admissable open covering $\Omega_{\mathbf{v},A}$, $\mathbf{v} \in A$ of the rigid analytic variety $\mathcal{Y}_A = \Omega_A$. We have a $T(K)$-equivariant reduction map $\psi_A : \Omega_A \to A \cong \mathbb{R}$, defined by $\psi_A(x) := 2v_K(x_1/x_2)$. Here v_K denotes the additive valuation of K normalised such that $v_K(\pi) = 1$ for a uniformising element π. The identification of the apartment A with the real numbers $\mathbb{R}$ is such that the verices correspond with the integers $\mathbb{Z} \subset \mathbb{R}$. The admissable covering $\Omega_{\mathbf{v},A}$, $\mathbf{v} \in A$, consists of the spaces $\Omega_{\mathbf{v},A} := \{x \in \Omega_A \mid d_A(\psi_A(x), \mathbf{v}) < 1\}$, since $\parallel \mathbf{v} \parallel_A = \{z \in A \mid d_A(z, \mathbf{v}) < 1\}$. By taking $\mathcal{Z}_{\mathbf{v},A} := \varphi_A^{-1}(\Omega_{\mathbf{v},A})$ one obtains an admissable covering of the variety $\mathcal{Z}_A$.

One needs suitable admissable open subspaces $\mathcal{Y}_{\mathbf{v}} \subset \mathcal{Z}_{\mathbf{v},A}$ for the general construction. These spaces correspond to the open subsets $\parallel \mathbf{v} \parallel_{\mathcal{B}_G} \subset \mathcal{B}_G$. Here we use the refined simplicial structure of the tree $\mathcal{B}_G$. One has for a vertex $\mathbf{v} \in |\mathcal{T}_\Gamma| \subset \mathcal{B}_G$ that $\parallel \mathbf{v} \parallel_{\mathcal{B}_G} = \{z \in |\mathcal{T}_\Gamma| \mid d_{\mathcal{B}_G}(z, \mathbf{v}) < 1\} \bigcup \{z \notin |\mathcal{T}_\Gamma| \mid d_{\mathcal{B}_G}(z, \mathbf{v}) < 1/2\}$. For the vertices $\mathbf{v} \notin |\mathcal{T}_\Gamma|$, i.e. the midpoints of the edges that are not contained in $|\mathcal{T}_\Gamma|$, one has $\parallel \mathbf{v} \parallel_{\mathcal{B}_G} = \{z \in \mathcal{B}_G| \mid d_{\mathcal{B}_G}(z, \mathbf{v}) < 1/2\}$. Then $\bigcup_{\mathbf{v} \in |\mathcal{T}_\Gamma|} \parallel \mathbf{v} \parallel_{\mathcal{B}_G} = \mathcal{B}_G - \{\mathbf{v} \mid \mathbf{v} \notin |\mathcal{T}_\Gamma|\}$. For any apartment $A \ni \mathbf{v}$ one has a well-defined space $\mathcal{Z}_{\mathbf{v},A} := \{x \in \mathcal{Z}_A \mid \psi_A(\varphi_A(x)) \in \| \mathbf{v} \|_A = A \bigcap \| \mathbf{v} \|_{\mathcal{B}_G}\}$. The space $\mathcal{Y}_{\mathbf{v}}$ is defined by $\mathcal{Y}_{\mathbf{v}} := \bigcap_{A \ni \mathbf{v}} \mathcal{Z}_{\mathbf{v},A} = \{x \in \bigcap_{A \ni \mathbf{v}} \mathcal{Z}_A \mid \forall (A \ni \mathbf{v}) \ \psi_A(\varphi_A(x)) \in \| \mathbf{v} \|_{\mathcal{B}_G}\}$. For the points $x \in \mathcal{Y}_{\mathbf{v}}$ there exists a map $\psi_{\mathbf{v},\mathcal{B}_G} : \mathcal{Y}_{\mathbf{v}} \to \| \mathbf{v} \|_{\mathcal{B}_G}$ such that for all apartments $A \subset \mathcal{B}_G$ that contain the point $\psi_{\mathbf{v},\mathcal{B}_G}(x)$ the equality $\psi_A(\varphi_A(x)) = \psi_{\mathbf{v},\mathcal{B}_G}(x)$ holds. The maps $\psi_{\mathbf{v},\mathcal{B}_G}$ for the vertices $\mathbf{v} \in |\mathcal{T}_\Gamma|$ glue together and give a map $\psi_{\mathcal{B}_G} : \bigcup_{\mathbf{v} \in \mathcal{B}_G} \mathcal{Y}_{\mathbf{v}} \to \mathcal{B}_G$. The map $\psi_{\mathcal{B}_G}$ is of course Drinfel'ds reduction map. Using the map $\psi_{\mathcal{B}_G}$ the admissable open subspace $\mathcal{Y}_{\mathbf{v}} \subset \mathcal{Z}_{\mathbf{v},A}$ can be described as consisting of the points $x \in \mathcal{Z}_{\mathbf{v},A}$ such that $\psi_{\mathcal{B}_G}(x)$ is well-defined and, moreover, $\psi_{\mathcal{B}_G}(x) \in \| \mathbf{v} \|_{\mathcal{B}_G}$. The subspace $\mathcal{Y}_{\mathbf{v}} \subset \mathcal{Z}_{\mathbf{v},A}$ does not depend on the apartment $A \ni \mathbf{v}$ that is contained in $\mathcal{T}_\Gamma$ that is used in the description above.

Let us now consider the situation around the vertices that are not contained in $|\mathcal{T}_\Gamma|$. Let $\mathbf{v}'$ be a vertex that is not contained in $|\mathcal{T}_\Gamma|$. Let $\mathbf{e} \in \mathcal{B}_G$ be an edge containing the vertex $\mathbf{v}'$. Here we use the refined simplicial structure of the tree $\mathcal{B}_G$. Let $A' \ni \mathbf{v}'$ be an apartment. Then $\mathbf{e} \in A'$. Let $\mathbf{v}$ be the vertex of $\mathbf{e}$ different from $\mathbf{v}'$. We choose an apartment $A \ni \mathbf{v}$ that is contained in transversal system $\mathcal{T}_\Gamma$. We assume that the apartments A and A' intersect transversally, i.e. that $A \bigcap A' = \{\mathbf{v}\}$ holds.

The analytic space $\mathcal{Y}_{\mathbf{v}'}$ is now defined as $\mathcal{Y}_{\mathbf{v}'} := \Omega_{\mathbf{v}',A'} \subset \Omega_{A'}$. The subspace $\mathcal{Y}^\vee_{\mathbf{e}} \subset \mathcal{Y}_{\mathbf{v}'}$ that corresponds to the edge $\mathbf{e} \ni \mathbf{v}, \mathbf{v}'$ is defined by $\mathcal{Y}^\vee_{\mathbf{e}} := \Omega_{\mathbf{e},A'}$. The admissable open subspace $\mathcal{Y}_{\mathbf{e}} \subset \mathcal{Y}_{\mathbf{v}}$ is defined by $\mathcal{Y}_{\mathbf{e}} := \{x \in \mathcal{Y}_{\mathbf{v}} \mid \psi_{\mathcal{B}_G}(x) \in \mathbf{e}, \psi_{\mathcal{B}_G}(x) \neq \mathbf{v}\}$.

To identify the spaces $\mathcal{Y}_{\mathbf{e}}$ and $\mathcal{Y}^\vee_{\mathbf{e}}$ with each other, we describe these spaces more explicitly. Since the apartments A and A' intersect transversally, the intersection $\Omega_A \bigcap \Omega_{A'}$ consists of a point a such that $\psi_A(a) = \psi_{A'}(a) = \mathbf{v}$. We may assume that A is the standard apartment, corresponding to the coordinates x_0, x_1, x_2 of $\mathbb{P}^2_K$. Let $a = (0, a_1, a_2)$. The set $\varphi_A^{-1}(a) \subset \mathcal{Z}_A$ consists of the two points $(\pm a_0, a_1, a_2)$ such that $a_0^2 + a_1 a_2 = 0$. The points (a_0, a_1, a_2) and $(-a_0, a_1, a_2)$ span a subspace $\mathbb{P}^1_K \subset \mathbb{P}^2_K$. This line $\mathbb{P}^1_K$ contains the space $\Omega_{A'}$ and $\Omega_{A'} = \mathbb{P}^1_K - \{(\pm a_0, a_1, a_2)\}$ holds. The points of $\Omega_{A'}$ are therefore of the form $y_1(a_0, a_1, a_2) + y_2(-a_0, a_1, a_2)$ with $y_1/y_2 \in K^*$. We may assume that $\mathcal{Y}^\vee_{\mathbf{e}}$ consists of the points $y \in \Omega_{A'}$ with $|\pi|^{1/2} < |y_1/y_2|^2 < 1$. The space $\mathcal{Y}_{\mathbf{e}} \subset \mathcal{Y}_{\mathbf{v}}$ consists of the points such that $|\pi|^{1/2} < |x_1/x_2 - a_1/a_2|^2 < 1$ and $|x_0/x_2 - a_0/a_2| < 1$. Note that the subset consisting of the points $x \in \mathcal{Y}_{\mathbf{v}}$ such that $|\pi|^{1/2} < |x_1/x_2 - a_1/a_2|^2 < 1$ equals the union $\mathcal{Y}_{\mathbf{e}} \bigcup \mathcal{Y}_{\mathbf{e}'}$, where $\mathbf{e}, \mathbf{e}' \in A'$ are the two edges containing the vertex $\mathbf{v}$. We can now identify a point $x \in \mathcal{Y}_{\mathbf{e}}$ and a point $y \in \mathcal{Y}^\vee_{\mathbf{e}}$ whenever $y_1/y_2 = x_1/x_2 - a_1/a_2$ holds. One verifies that this is well-defined and gives a bijection $\mathcal{Y}_{\mathbf{e}} \cong \mathcal{Y}^\vee_{\mathbf{e}}$. This identification can be used to glue the spaces $\mathcal{Y}_{\mathbf{v}}$ and $\mathcal{Y}_{\mathbf{v}'}$. The identifications $\mathcal{Y}_{\mathbf{e}} \cong \mathcal{Y}_{\mathbf{e}}^{\vee}$ for all edges $\mathbf{e} \in \mathcal{B}_G - |\mathcal{T}_\Gamma|$ can be done in a Γ-equivariant way. Hence one has obtained a rigid analytic variety $\mathcal{Y}_\Gamma$ on which the group Γ acts discretely with proper quotient.

Let $\mathcal{Y}_\Gamma^\circ := \bigcup_{\mathbf{v}\in|\mathcal{T}_\Gamma|} \mathcal{Y}_{\mathbf{v}}$ and let $\mathcal{Y}_\Gamma := \bigcup_{\mathbf{v}\in\mathcal{B}_G} \mathcal{Y}_{\mathbf{v}}$. The group Γ acts discretely on the analytic spaces $\mathcal{Y}_\Gamma^\circ$ and $\mathcal{Y}_\Gamma$. One has that $\psi_{\mathcal{B}_G}(\mathcal{Y}_\Gamma^\circ) = \mathcal{B}_G(\mathbb{Q}) - \{\mathbf{v} \in \mathcal{B}_G \mid \mathbf{v} \notin |\mathcal{T}_\Gamma|\}$. The quotient $\mathcal{Y}_\Gamma^\circ/\Gamma$ is proper if and only if the transversal system $\mathcal{T}_\Gamma$ is complete. In that case $\mathcal{Y}_\Gamma^\circ \cong \Omega_{1,K} := \mathbb{P}_K^1 - \mathbb{P}^1(K)$.

In general, $\mathcal{Y}_\Gamma^\circ = \Omega_{1,K} - \{x \in \Omega_{1,K} \mid \psi_{\mathcal{B}_G}(x) = \mathbf{v}, \mathbf{v} \in \mathcal{B}_G - |\mathcal{T}_\Gamma|\}$ holds. By construction the quotient $\mathcal{Y}_\Gamma/\Gamma$ is proper. In all cases the analytic variety $\mathcal{Y}_\Gamma$ is isomorphic to the p-adic upper half plane $\mathcal{Y}_\Gamma \cong \Omega_{1,K} = \mathbb{P}_K^1 - \mathbb{P}^1(K)$. Hence we have obtained the uniformisation of Mumford curves by $\Omega_{1,K}$.

11.19. Symmetric spaces over the real numbers. Usually one considers the affine building of a p-adic linear algebraic group $G(K)$ to be its symmetric space in a metric sense. If the characteristic of the local field K is zero, then all discrete arithmetic subgroups $\Gamma \subset G(K)$ of finite co-volume are co-compact. Therefore compactifying a symmetric space (i.e. the affine building) does not seem useful for uniformisation purposes.

Our approach is to interpretate the affine building as the p-adic analog of a Γ-equivariant compactification of a symmetric space. The question then becomes to find the symmetric space of which the building $\mathcal{B}_G$ is the Γ-equivariant compactification. This we define to be the space $\| \mathcal{T}_\Gamma \|_{\mathcal{B}_G} = \mathcal{B}_G - |B_1(\mathcal{T}_\Gamma)|$, where $\mathcal{T}_\Gamma$ is a Γ-invariant almost complete transversal system of $H(K)$-subbuildings. The subspace $\| \mathcal{T}_\Gamma \|_{\mathcal{B}_G} \subset \mathcal{B}_G$ consists of infinitely many connected components, unless $\| \mathcal{T}_\Gamma \|_{\mathcal{B}_G} = \mathcal{B}_G$ holds. The simplices in the complement $|B_1(\mathcal{T}_\Gamma)|$ of the thus constructed "symmetric space" $\| \mathcal{T}_\Gamma \|_{\mathcal{B}_G}$ form a union of buildings of lower rank, after possibly refining the simplicial structure of the building $\mathcal{B}_G$.

This point of view has of course the consequence that the Γ-equivariant compactification is the affine building of the linear algebraic group $G(K)$ and does not depend at all on the arithmetic group Γ. The "symmetric space" itself, however, depends very much on the arithmetic group.

The boundary $|B_1(\mathcal{T}_\Gamma)|$ is not unique. By subdividing some simplices that are not contained in $|\mathcal{T}_\Gamma|$ one can obtain a different boundary. In this case the compactification of $\| \mathcal{T}_G \|_{\mathcal{B}_G}$ remains the building $\mathcal{B}_G$, but the simplicial structure of the building has been refined somewhat.

The construction provides the affine building $\mathcal{B}_G$ with some extra structure in the form of a Γ-invariant almost complete transversal system $\mathcal{T}_\Gamma$ and possibly a refined simplicial structure. In fact, as a compactification of $\| \mathcal{T}_\Gamma \|_{\mathcal{B}_G}$ the space $\mathcal{B}_G = \| \mathcal{T}_\Gamma \|_{\mathcal{B}_G} \bigcup |B_1(\mathcal{T}_\Gamma)|$ has some resemblance to the Satake- and Baily-Borel compactifications of a hermitian symmetric space.

There are two situations where the Γ-invariant transversal system $\mathcal{T}_\Gamma$ is such that $|\mathcal{T}_\Gamma| = \mathcal{B}_G$:

i) The group G is a K-split group with root system Φ_G is of type A_r, D_r, E_6, E_7 or E_8. Then $\mathcal{T}_\Gamma = \{\mathcal{B}_G\}$ is trivial.

ii) The transversal system $\mathcal{T}_\Gamma$ is non-trivial and $\bigcup_{\mathbf{b}\in\mathcal{T}_\Gamma} \mathbf{b} = \mathcal{B}_G$.

In these cases the quotient space $\| \mathcal{T}_\Gamma \|_{\mathcal{B}_G} /\Gamma = \mathcal{B}_G/\Gamma$ is compact. If $\| \mathcal{T}_\Gamma \|_{\mathcal{B}_G} \neq \mathcal{B}_G$, then the quotient $\| \mathcal{T}_\Gamma \|_{\mathcal{B}_G} /\Gamma = (\mathcal{B}_G - |B_1(\mathcal{T}_\Gamma)|)/\Gamma$ is non-compact.

In general there does not seem to be a rigid analytic variety $\mathcal{Y}_\Gamma$ associated to the "symmetric space" $\| \mathcal{T}_\Gamma \|_{\mathcal{B}_G}$ that is such that the quotient $\mathcal{Y}_\Gamma/\Gamma$ can be compactified and is algebraizable. This can only be done if the group $H(K)$ acts on a rigid space $\mathcal{Y}_H$, that is such that the quotients $\mathcal{Y}_H/\Gamma_H$ by discrete co-compact subgroups $\Gamma_H \subset H(K)$ are proper algebraic varieties. In particular, this holds if the $H(K)$ has root system Φ_H of type A. Then the space $\mathcal{Y}_H$ is Drinfeld's symmetric space Ω_H.

11.20. Shimura varieties. We have only proved fully the ideas sketched above for the group $SU(3, L)$. In that case our construction corresponds with the uniformisation of complex curves by arithmetic subgroups of $SL(2, \mathbb{R})$. Then the uniformising space is a hermitian symmetric space. This is the complex unit ball. The quotient of this symmetric space by the arithmetic group is either compact or non-compact, depending on whether the arithmetic group is co-compact or has only finite co-volume. If the quotient is non-compact it can always be compactified.

In essence we view the Deligne-Lusztig variety $X(w)_G$ for a Coxeter element w of the Weyl group of $G(k)$ as an algebraic geometric variety that represents the spherical building of $G(k)$. Lifting the Deligne-Lusztig variety to characteristic zero and using transversal systems gives us a realisation of the affine building of $G(K)$ as a rigid analytic variety on which the discrete group $\Gamma \subset G(K)$ acts discretely. The analytic varieties depend on the existence of Γ-invariant transversal sytems and suitable properties of Deligne-Lusztig varieties. Such a construction neither requires the existence of hermitian symmetric spaces over the field of the real numbers $\mathbb{R}$, nor does it relate to any specific moduli problem. So let us compare these ideas with some known properties of Shimura varieties.

A moduli problem usually involves an arithmetic group G defined over the rational field $\mathbb{Q}$. Therefore $G \otimes \mathbb{C}$ and $G \otimes \mathbb{C}_p$ have the same root sytem. Let us assume that the Shimura variety involved has totally degenerate reduction over the p-adics, meaning that it has a p-adic uniformisation. The Shimura variety is then uniformised over both the real and the p-adic numbers using an arithmetic subgroup of the group $G \otimes \mathbb{C}$ and $G \otimes \mathbb{C}_p$, respectively.

In our case the uniformising spaces $\mathcal{Y}_\Gamma$ are modulo a finite étale map completely determined by the group H. Locally the quotient looks like a finite étale covering of a quotient $\mathcal{Y}_H/(\Gamma \bigcap H)$. Let us assume that the quotient $\mathcal{Y}_\Gamma/\Gamma$ can be defined over some number field and therefore possibly has a uniformisation over the real numbers $\mathbb{R}$. In that case, I would expect that this real uniformisation only depends on the group $H(K) \supset \Gamma \bigcap H(K)$ that is used to construct the transversal system $\mathcal{T}_\Gamma$ of the building $\mathcal{B}_G$ and not at all on the particular group $G(K) \supset \Gamma$.

Over the real numbers a Shimura variety can be described as the quotient of a hermitian symmetric space by an arithmetic group. A symmetric space is always an open analytical subspace of some projective homogeneous variety X_{λ_G}, where λ_G is a suitable fundamental weight for the root system Φ_G. The varieties $\mathcal{Y}_\Gamma \subset X_{\lambda_G}$ sketched above have some codimension in the space X_{λ_G}. Therefore they are not open analytical subspaces. The only exception is Drinfeld's symmetric space $\Omega_r \subset \mathbb{P}^r_K$.

The ideas sketched here also seem to be at odds with the results on moduli spaces by Rapoport, Zink et al.(See [RZ] and [Ra]). They constructed a map from a certain moduli space to a period domain that is actually an open analytical subspace of a projective homogeneous variety $X_{\lambda_G} = G/P_{\lambda_G}$. This moduli space is closely related to Shimura varieties.

Both our construction and the one of Rapoport and Zink generalise Drinfeld's symmetric space. We only use a rigid analytic description of an étale covering of this space, whereas Rapoport and Zink use the moduli interpretation. However, this should not result in different spaces. Therefore the difference must be caused by the fact that we use a different Frobenius map than Drinfeld seems to do in his construction (See §5.13 above). Instead of using the standard Frobenius action F that fixes each connected component of the system of étale coverings, we use a twist wF of the standard Frobenius F by an element w of the Weyl group. The Frobenius wF does permute connected components of the system of étale coverings of Drinfeld's symmetric space.

The discrepancy between what is known about Shimura varieties and the quotients of the analytic spaces sketched here is surprisingly large. On the other hand for groups G such that the group H is of type A, the spaces $\mathcal{Y}_\Gamma$ are locally finite étale coverings of Drinfeld's symmetric space and some connection with Shimura varieties seems plausible.

References

[BGR] S. Bosch, U. Güntzer and R. Remmert, *Non-archimedean Analysis*, Springer Verlag, 1984.

[D] V.G. Drinfel'd, *Coverings of p-Adic Symmetric Regions*, Functional Anal. Appl. **10** (1976), 107-115.

[FP] J. Fresnel and M. van der Put, *Geométrie analytique rigide et applications*, Progres in Math. **18**, Birkhäuser, 1981.

[GP] L. Gerritzen and M. van der Put, *Schottky groups and Mumford curves*, Springer Lecture Notes in Math. **817** (1980).

[H] D.W. Hoffmann, *On positive definite hermitian forms*, Manuscripta Math. **71** (1991), 399-429.

[LV] K.F. Lai and H. Voskuil, *P-adic automorphic functions for the unitary group in three variables*, Algebra Colloquium **7** (2000), 335-360.

[OR] S. Orlik and M. Rapoport, *Deligne-Lusztig varieties and period domains over finite fields*, J. Algebra **320(3)** (2008), 1220-1234.

[PV] M. van der Put and H. Voskuil, *Symmetric spaces associated to split algebraic groups over a local field*, J. Reine Angew. Math. **433** (1992), 69-100.

[Ra] M. Rapoport, *A guide to the reduction modulo p of Shimura varieties*, Astèrisque **298** (2005), 271-318.

[RZ] M. Rapoport and T. Zink, *Period spaces for p-divisible groups*, Ann. Math. Stud. **141**, Princeton University Press, 1996.

[Ro] F. Rodier, *Nombres de points des surfaces de Deligne et Lusztig*, J. Algebra **227(2)** (2000), 706-766.

[S] A. Schiemann, *Classification of hermitian forms with the neighbourhood method*, J. Symb. Comput. **26** (1998), 487-508.

[Te] J. Teitelbaum, *Geometry of an étale covering of the p-adic upper half plane*, Ann. Inst. Fourier (Grenoble) **40** (1990), 69-78.

[V] H. Voskuil, *On the action of the unitary group on the projective plane over a local field*, J. Austral. Math. Soc. (Series A) **62** (1997), 371-397.